Differentiation

Table of standard derivatives

$f(x)$	$f'(x)$
$\tan x$	$\sec^2 x$ or $\left(\frac{1}{\cos^2 x}\right)$
$\operatorname{cosec} x \quad \left(\frac{1}{\sin x}\right)$	$-\operatorname{cosec} x \cot x$
$\sec x \quad \left(\frac{1}{\cos x}\right)$	$\sec x \tan x$
$\cot x \quad \left(\frac{1}{\tan x}\right)$	$-\operatorname{cosec}^2 x$
$\sin^{-1} x \quad \arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x \quad \arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x \quad \arctan x$	$\frac{1}{1+x^2}$
$\ln x$	$\frac{1}{x}$
e^x	e^x

You are expected to already know that

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$

Lesson 1 - Using the Chain Rule to differentiate composite functions

From higher maths you should be aware that a composite function in the form $h(x) = g(f(x))$ has the derivative $h'(x) = g'\big(f(x)\big) \times f'(x)$

Changing to Leibniz' notation, the composite function is expressed as dependent variables – i.e. $y = u$ where $u = x$.

The chain rule states that the derivative is $\boldsymbol{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$

Example 1 $\quad y = (2x^3 - 1)^4 \qquad y = u^4, \quad u = 2x^3 - 1$

$\frac{dy}{du} = 4u^3 \qquad \frac{du}{dx} = 6x^2$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{dy}{dx} = 4u^3 \times (6x^2)$$
$$= 4(2x^3 - 1)^3 6x^2$$
$$= \mathbf{24x^2(2x^3 - 1)^3}$$

Example 2 $\quad y = \cos(\sin x) \qquad y = \cos u, \quad u = \sin x$

$\frac{dy}{du} = -\sin u \qquad \frac{du}{dx} = \cos x$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{dy}{dx} = -\sin u \times \cos x$$
$$= -\boldsymbol{\sin(\sin x)\cos x} \qquad 0 \le x \le 2\pi$$

Example 2 $\quad y = (1 - \cos(3x))^2 \qquad y = u^2, \quad u = 1 - \cos v, \; v = 3x$

$\frac{dy}{du} = 2u, \qquad \frac{du}{dv} = \sin v, \qquad \frac{dv}{dx} = 3$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$

$$\frac{dy}{dx} = 2u \times (\sin v) \times 3$$
$$= 2(1 - \cos v) \times 3\sin(v)$$
$$= \mathbf{6\sin(3x)(1 - \cos(3x))}$$

In your MIA textbook – Exercise 4.3 Q1 – 5 and Exercise 4.4 – Q1-3 only

In Leckie and Leckie - Exercise 2A, Q1 – 3

Lesson 2 - The Product Rule

For a function in the form $h(x) = f(x) \times g(x)$ $\qquad \boldsymbol{h = f \times g}$

The derivative is $\boldsymbol{h'(x) = f'(x) \times g(x) + f(x) \times g'(x)}$ $\qquad \boldsymbol{h' = f'g + fg'}$

Example 1 $\quad h(x) = x^7 \sin 3x,$ $\qquad f(x) = x^7,$ $\quad g(x) = \sin 3x$

$f'(x) = 7x^6,$ $\quad g'(x) = 3\cos 3x$

$\boldsymbol{h'(x) = 7x^6 \sin 3x + 3x^7 \cos 3x}$

Example 2 $\quad h(x) = \sin x \cos x,$ $\quad f(x) = \sin x,$ $\quad g(x) = \cos x$

$f'(x) = \cos x,$ $\quad g'(x) = -\sin x$

$$\begin{aligned} h'(x) &= \cos x \cos x - \sin x \sin x \\ &= \cos^2 x - \sin^2 x \\ &= \boldsymbol{\cos 2x} \end{aligned}$$

From higher maths

Watch out for examples which also contain composite functions

Example 3 $\quad h(x) = x(2x+1)^4$ $\qquad f(x) = x,$ $\quad g(x) = (2x+1)^4$

$f'(x) = 1,$ $\quad g'(x) = 8(2x+1)^3$

$$\begin{aligned} h'(x) &= 1(2x+1)^4 + 8x(2x+1)^3 \\ &= (2x+1)^3[(2x+1) + 8x] \\ &= \boldsymbol{3(10x+1)(2x+1)^3} \end{aligned}$$

Where possible give your answer in the simplest form

A useful resource is https://www.derivative-calculator.net/.

This allows you to check your differentiation and your algebraic manipulation

In your MIA textbook – Exercise 4.5 - All of Q1 and 2, some of Q3 and 4, one of Q5 or 6
In Leckie and Leckie - Exercise 2C- do the whole exercise

Lesson 3 - The Quotient Rule

For a function in the form $h(x) = \frac{f(x)}{g(x)}$ $\qquad$ $\boldsymbol{h = \frac{f}{g}}$

The derivative is $\boldsymbol{h'(x) = \frac{f'(x)\times g(x) - f(x)\times g'(x)}{(g(x))^2}}$ $\qquad$ $\boldsymbol{h' = \frac{f'g - fg'}{g^2}}$

Example 1 $\qquad h(x) = \frac{5x}{x^2+1}$ $\qquad f(x) = 5x, \quad g(x) = x^2 + 1$

$f'(x) = 5, \quad g'(x) = 2x$

$$h'(x) = \frac{5 \times (x^2 + 1) - 5x(2x)}{(x^2 + 1)^2}$$

$$h'(x) = \frac{5x^2 + 5 - 10x^2}{(x^2 + 1)^2}$$

$$\boldsymbol{h'(x) = \frac{5 - 5x^2}{(x^2 + 1)^2}}$$

Example 2 $\qquad h(x) = \frac{(x+1)^2}{x^3}$ $\qquad f(x) = (x + 1)^2, \quad g(x) = x^3$

$f'(x) = 2(x + 1), \quad g'(x) = 3x^2$

$$h'(x) = \frac{2(x + 1) \times x^3 - (x + 1)^2 \times 3x^2}{(x^3)^2}$$

$$h'(x) = \frac{2x^3(x + 1) - 3x^2(x + 1)^2}{x^6}$$

$$\boldsymbol{h'(x) = \frac{2x(x + 1) - 3(x + 1)^2}{x^4}}$$

Example 3

$h(x) = \tan x = \frac{\sin x}{\cos x}$

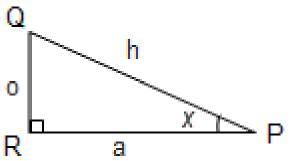
$$h'(x) = \frac{\cos x \cos x - \sin x \times (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \qquad \cos^2 x + \sin^2 x = 1$$

$$= \boldsymbol{\frac{1}{\cos^2 x}}$$

In your MIA textbook – Exercise 4.6 - All of Q1, some of Q2 and 3, one of Q4 or 5, one of Q6 or 7 for extension. Exercise 4.7 is a summary of the chain, product, and quotient rules.
In Leckie and Leckie - Exercise 2D, Q1,2,3 and 5

Lesson 4 – 6 trig functions and their derivatives



For angle x, there are six functions where each function is the **ratio** of two sides of the triangle. The only difference between the six functions is which pair of sides we use.

The three primary trig functions are $\sin x = \frac{o}{h}$ $\cos x = \frac{a}{h}$ $\tan x = \frac{o}{a}$

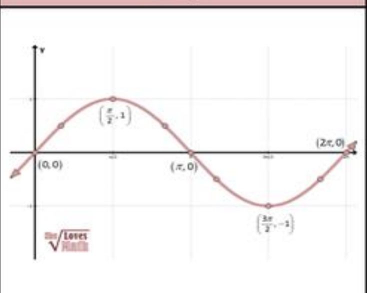
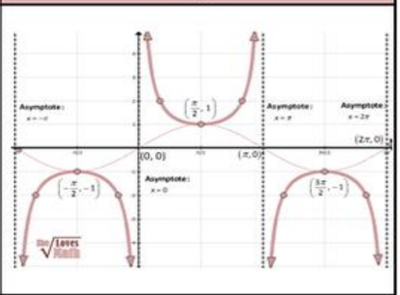
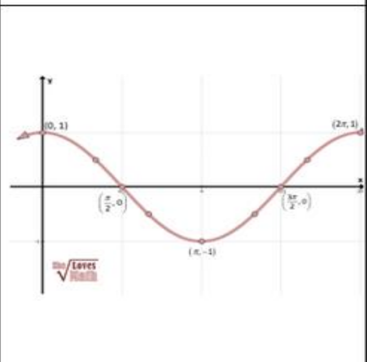
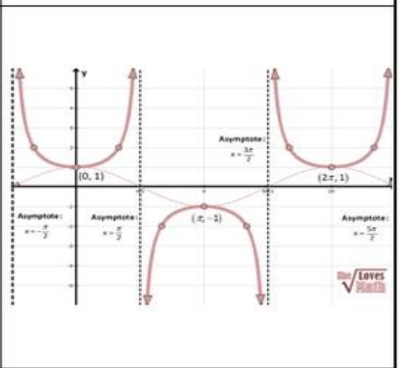
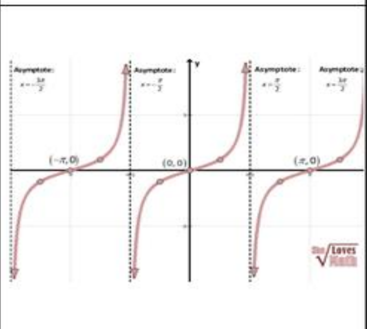
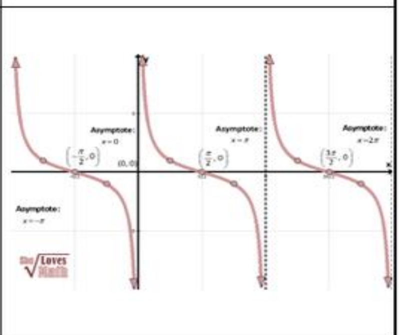
The next three functions are the reciprocals of sin, cos and tan

Cosecant $\operatorname{cosec} x = \frac{1}{\sin x} = \frac{h}{o}$ Secant $\sec x = \frac{1}{\cos x} = \frac{h}{a}$

Cotangent $\cot x = \frac{1}{\tan x} = \frac{a}{o}$ where $\sin x \neq 0, \cos x \neq 0 \tan x \neq 0$

Note $\operatorname{cosec} x \neq \sin^{-1} x$ and the same holds for $\sec x$ and $\cot x$

Graphs of the six trig functions – *just in case you are interested*

Parent Function	Graph	Parent Function	Graph
$y = \sin(x)$ Odd **Domain:** $(-\infty, \infty)$ **Range:** $[-1, 1]$ **Period:** 2π **Zeros:** $(\pi k, 0)$, $k \in$ Integers		$y = \csc(x)$ Odd **Domain:** $x \neq \pi k$ **Range:** $(-\infty, -1] \cup [1, \infty)$ **Asymptotes:** $x = \pi k$ **Period:** 2π **Zeros:** None	
$y = \cos(x)$ Even **Domain:** $(-\infty, \infty)$ **Range:** $[-1, 1]$ **Period:** 2π **Zeros:** $\left(\frac{\pi}{2} + \pi k, 0\right)$		$y = \sec(x)$ Even **Domain:** $x \neq \frac{\pi}{2} + \pi k$ **Range:** $(-\infty, -1] \cup [1, \infty)$ **Asymptotes:** $x = \frac{\pi}{2} + \pi k$ **Period:** 2π **Zeros:** None	
$y = \tan(x)$ Odd **Domain:** $x \neq \frac{\pi}{2} + \pi k$ **Range:** $(-\infty, \infty)$ **Asymptotes:** $x = \frac{\pi}{2} + \pi k$ **Period:** π **Zeros:** $(\pi k, 0)$		$y = \cot(x)$ Odd **Domain:** $x \neq \pi k$ **Range:** $(-\infty, \infty)$ **Asymptotes:** $x = \pi k$ **Period:** π **Zeros:** $\left(\frac{\pi}{2} + \pi k, 0\right)$	

Example 1 $h(x) = \cot x$

Using the quotient rule

$$h(x) = \frac{f(x)}{g(x)}$$

$$f(x) = \cos x$$
$$f'(x) = -\sin x$$

$$g(x) = \sin x$$
$$g'(x) = \cos x$$

$$h(x) = \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

$$h'(x) = \frac{-\sin x\ \sin x - \cos x\ \cos x}{\sin^2 x}$$

$$h'(x) = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$h'(x) = \frac{-1}{\sin^2 x}$$

$$\boldsymbol{h'(x) = -\mathbf{cosec}^2 x}$$

Example 2 $f(\theta) = \sec 7\theta$

Using the chain rule

$y = \sec u, \quad u = 7\theta$

$$\frac{dy}{du} = \sec u \tan u, \quad \frac{du}{dx} = 7$$

$$f'(\theta) = \frac{dy}{du} \times \frac{du}{d\theta}$$

$$f'(x) = \sec u \tan u \times 7$$

$$\boldsymbol{f'(x) = 7\sec 7\theta \tan 7\theta}$$

Example 3 $y = \cot(x^2)$

Using the chain rule

$y = \cot u, \quad u = x^2$

$$\frac{dy}{du} = -\mathrm{cosec}^2 u, \quad \frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = -\mathrm{cosec}^2 u \times 2x$$

$$\boldsymbol{\frac{dy}{dx} = = -2x\mathbf{cosec}^2(x^2)}$$

Example 4 $y = \tan(\sin x)$

Using the chain rule

$y = \tan u, \quad u = \sin x$

$$\frac{dy}{du} = \sec^2 u, \quad \frac{du}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \sec^2 u \times \cos x$$

$$\boldsymbol{\frac{dy}{dx} = \mathbf{sec}^2(\mathbf{sin}\, x)\,\mathbf{cos}\, x}$$

Be aware that when working with trig there can be more than one correct answer,

for example - $2\cot x$ can also be written as $\frac{2}{\tan x}$ or $2\left(\frac{\cos x}{\sin x}\right)$ etc

In your MIA textbook – Exercise 4.8 - Q1 answered, do Q2 and Q5 & 6. If attempting Q4 do a – c.
In Leckie and Leckie - Exercise 2F

Lesson 5 – Exponential and log functions

$f(x)$	$f'(x)$
e^x	e^x
$\ln x$	$\frac{1}{x}$

Use the chain, product and quotient rules to differentiate log and exp functions

Exponential ex 1 $y = e^{7x}$, $y = e^u$, $u = 7x$ $\frac{dy}{du} = e^u$, $\frac{du}{dx} = 7$

$\frac{dy}{dx} = 7e^u = \boldsymbol{7e^{7x}}$

Exponential ex 2 $y = e^{(3x^2+1)}$, $y = e^u$, $u = 3x^2 + 1$ $\frac{dy}{du} = e^u$, $\frac{du}{dx} = 6x$

$\frac{dy}{dx} = 6xe^u = \boldsymbol{6xe^{3x^2+1}}$

Exponential ex 3 $y = e^{\sin x}$, $y = e^u$, $u = \sin x$ $\frac{dy}{du} = e^u$, $\frac{du}{dx} = \cos x$

$\frac{dy}{dx} = \cos x\, e^u = \boldsymbol{\cos x\, e^{\sin x}}$

Log example 1 $y = \ln 3x$, $y = \ln u$, $u = 3x$ $\frac{dy}{du} = \frac{1}{u}$, $\frac{du}{dx} = 3$

$\frac{dy}{dx} = \frac{1}{u} \times 3 = \frac{3}{3x} = \boldsymbol{\frac{1}{x}}$

Log example 2 $y = \ln(2x + 1)$, $y = \ln u$, $u = 2x + 1$ $\frac{dy}{du} = \frac{1}{u}$, $\frac{du}{dx} = 2$

$\frac{dy}{dx} = \frac{1}{u} \times 2 = \boldsymbol{\frac{2}{2x+1}}$

Log example 3 $y = \tan(\ln x)$, $y = \tan u$, $u = \ln x$ $\frac{dy}{du} = \sec^2 u$, $\frac{du}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \sec^2 u \times \frac{1}{x} = \boldsymbol{\frac{\sec^2(\ln x)}{x}}$

When working with an exponential base other than *e* such as $y = 3^x$,

Take natural logs of both sides	$\ln y = \ln 3^x$
Use laws of logs $\ln m^n = n \ln m$	$\ln y = x \ln 3$
Return to exponential form using base e. Note that $\ln 3$ is a constant	$y = e^{x \ln 3}$
Differentiate using the chain rule	$\frac{dy}{dx} = e^{x \ln 3} \times \ln 3$
Use laws of logs $n \ln m = \ln m^n$	$\frac{dy}{dx} = \left(e^{\ln 3^x}\right) \ln 3$
Use the identity $a^{\log_a n} = n$	$\frac{dy}{dx} = (3^x) \ln 3$
Thus the derivative of $y = 3^x$ is	$\frac{dy}{dx} = \ln 3\,(3^x)\ \ or\ \ \ln 3 \cdot 3^x$

When working with $\log_a$ rather than natural logs such as $y = \log_7 x$

Change log base to natural logs using $\log_a b = \ln b \times \frac{1}{\ln a}$	$y = \ln x \times \frac{1}{\ln 7}$
Differentiate	$\frac{dy}{dx} = \frac{1}{x} \times \frac{1}{\ln 7} = \boldsymbol{\frac{1}{x \ln 7}}$

Example - Differentiate $y = \log_3(5x + 2)$

Change to natural logs	$y = \ln(5x + 2) \times \frac{1}{\ln 3}$
Differentiate	$\frac{dy}{dx} = \frac{1}{5x + 2} \times 5 \times \frac{1}{\ln 3} = \boldsymbol{\frac{5}{(5x + 2) \ln 3}}$

Remember that you can always check your answers using https://www.derivative-calculator.net/

Also note that derivatives such as these are usually found by computers!

In your MIA textbook – Exercise 4.9 Q1 and Q2 should be enough, can also do Q3 and 4. Omit Q5
In Leckie and Leckie - Exercise 2B

Lesson 6 – Higher derivatives $f''(x), f'''(x)$ or $\frac{d^3y}{dx^3}$

Any differentiable function f can have a succession of derivatives

f' or $\frac{dy}{dx}$ is the first derivative, f'' or $\frac{d^2y}{dx^2}$ is the second derivative

f''' or $\frac{d^3y}{dx^3}$ is the third derivative, f^4 or $\frac{d^4y}{dx^4}$ is the fourth derivative, and so on

This process continues until f^n is a constant term

When you need to find higher derivatives, you must cycle through the lower derivatives to get the answer as there are no short cuts!

Example 1

$f\ (x) = x^3, \quad f'(x) = 3x^2, \quad f''(x) = 6x, \quad f'''(x) = 6 \quad f^4(x) = 0$

Note when $f(x) = x^n,\ f^n(x) = n! \ \rightarrow f(x) = x^3,\ f^3(x) = 3! = 6$

Example 2

$f(x) = \sin(2x) \qquad f'(x) = 2\cos(2x) \qquad f''(x) = -4\cos(2x)$

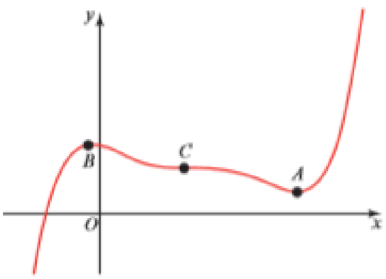
Example 3

$f(x) = e^{3x} \quad f'(x) = 3e^{3x} \quad f''(x) = 9e^{3x}, \quad f'''(x) = 27e^{3x} \quad f^4(x) = 81e^{3x}$

Example 4

$f(x) = \ln(2x+1), \quad f'(x) = \frac{2}{2x+1}, \quad f''(x) = \frac{-4}{(2x+1)^2}, \quad f'''(x) = \frac{-16}{(2x+1)^3}$ and so on

In the Properties of Functions unit we will use the second derivative test to identify the nature of a stationary point. For the function f where the graph $y = f(x)$ is show below.



When $f''(x) > 0$, the stationary point is a minimum (point A).

When $f''(x) < 0$, the stationary point is a maximum (point B).

When $f''(x) = 0$, the stationary point is a point of inflexion (point C)

In your MIA textbook – Exercise 4.10 – Q1,2,3 and 5. Review Ex for Chapter 4 is.
In Leckie and Leckie - Exercise 2E Q1, 2, 3 and 4.

Lesson 7 – Inverse functions $\sin^{-1} x, \cos^{-1} x$ and $\tan^{-1} x$

inverse function	derivative
$\sin^{-1} x$, arcsin x	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$, arccos x	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$, arctan x	$\dfrac{1}{x^2+1}$

Using the rule $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$ and the Pythagorean identity $\sin^2 x + \cos^2 x = 1$

Example 1 $y = \sin^{-1} x$, so $x = \sin y$ and $\dfrac{dx}{dy} = \cos y$ and $\cos y = \sqrt{1-\sin^2 y}$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \mathbf{\frac{1}{\sqrt{1-x^2}}} \qquad \text{as} \quad \sin y = x$$

Example 2 $y = \tan^{-1}(x+1)$, $y = \tan u$, $u = x+1$

$$\frac{dy}{dx} = \frac{1}{u^2+1} \times 1 = \frac{1}{(x+1)^2+1} = \frac{1}{x^2+2x+2}$$

Example 3 $y = \cos^{-1} e^{2x}$, $y = \cos^{-1} u$ where $u = e^z$ and $z = 2x$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-u^2}} \times 2e^z = -\frac{1}{\sqrt{1-e^{2z}}} \times 2e^{2x} = -\frac{2e^{2x}}{\sqrt{1-e^{4x}}}$$

Example 4 $y = \sin^{-1}\left(\frac{1}{x}\right)$, $y = \sin^{-1} u$ where $u = x^{-1}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \times -x^{-2} = \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \times -\frac{1}{x^2} = \frac{1}{\sqrt{1-\frac{1}{x^2}}} \times -\frac{1}{x^2}$$

$$= \frac{1}{\sqrt{\frac{x^2-1}{x^2}}} \times -\frac{1}{x^2} = \frac{\sqrt{x^2}}{\sqrt{x^2-1}} \times -\frac{1}{x^2} = \frac{x}{\sqrt{x^2-1}} \times -\frac{1}{x^2} = -\mathbf{\frac{1}{x\sqrt{x^2-1}}}$$

In your MIA textbook – Exercise 6.2 - Q1 to3 is sufficient, Q 4 onwards are not necessary
In Leckie and Leckie - Exercise 2G

Lesson 8 - Implicit differentiation

Up until now we have looked at functions which express y **explicitly** in terms of x, such as $y = \sin x,\ y = e^{2x}$.

For functions expressed **implicitly** such as $x^2 + y^2 = 25$ and $\ln y = x^2$ we use a special case of the chain rule to differentiate these.

Example 1 Differentiate $x^2 + y^2 = 25$

differentiate both sides with respect to x

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

differentiate x terms as normal,
differentiate y terms and multiply by $\frac{dy}{dx}$

$$2x + 2y\frac{dy}{dx} = 0$$

Rearrange into the form $\frac{dy}{dx} = \ \ldots$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{2y}$$

Express your derivative in the simplest form

$$\frac{dy}{dx} = -\frac{x}{y}$$

Example 2 Differentiate $\ln y = x^2$

differentiate both sides with respect to x

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x^2)$$

differentiate x terms as normal,
differentiate $\ln y$ as $\frac{1}{y}\frac{dy}{dx}$

$$\frac{1}{y} \times \frac{dy}{dx} = 2x$$

multiply through by y to find the derivative

$$\frac{dy}{dx} = 2xy$$

Example 3 differentiate $y^2 + x^3 - y^3 + 6 = 3y$

$$\frac{d}{dx}(y^2) + \frac{d}{dx}(x^3) - \frac{d}{dx}(y^3) + \frac{d}{dx}(6) = \frac{d}{dx}(3y)$$

$$2y\frac{dy}{dx} + 3x^2 - 3y^2\frac{dy}{dx} + 0 = 3\frac{dy}{dx}$$

Collect $\frac{dy}{dx}$ terms

$$3x^2 = 3\frac{dy}{dx} - 2y\frac{dy}{dx} + 3y^2\frac{dy}{dx}$$

$$3x^2 = (3 - 2y + 3y^2)\frac{dy}{dx}$$

Rearrange

$$\boldsymbol{\frac{dy}{dx} = \frac{3x^2}{3-2y+3y^2}}$$

Example 4 differentiate $\sin y + x^2y^3 - \cos x \ = 2y$

$$\frac{d}{dx}(\sin y) + \frac{d}{dx}(x^2y^3) - \frac{d}{dx}(\cos x) \ = \frac{d}{dx}(2y)$$

Use the product rule $\cos y\frac{dy}{dx} + \left(2xy^3 + \ x^2 3y^2\frac{dy}{dx}\right) = 2\frac{dy}{dx}$

Collect $\frac{dy}{dx}$ terms $2xy^3 + \sin x = 2\frac{dy}{dx} - \cos y\frac{dy}{dx} \ -3x^2y^2\frac{dy}{dx}$

$$2xy^3 = (2 - \cos y - 3x^2y^2)\frac{dy}{dx}$$

Rearrange $\boldsymbol{\frac{dy}{dx} = \ \frac{2xy^3}{2-\cos y-3x^2y^2}}$

Note:

- Differentiate x terms as normal
- Differentiate y terms and multiply by $\frac{dy}{dx}$
- Remember to use the product, quotient and chain rules when differentiating
- Collect x terms on one side and $\frac{dy}{dx}$ terms on the other
- Take out a common factor of $\frac{dy}{dx}$
- Express the derivative as $\frac{dy}{dx} =$
- Derivative can contain x and y terms, so if substituting to find the gradient of an implicit function you need to substitute for x and y

For example, the circle $x^2 + y^2 = 25$ has a derivative of $\frac{dy}{dx} \ = \ -\frac{x}{y}$

The gradient of the circle at that point (3,4) is $m = \ -\frac{3}{4}$

The gradient of the circle at point (0,5) is $m = \ -\frac{0}{5} = 0$

In your MIA textbook – Exercise 6.4 Q1 and 3 are sufficient, any of Q13, 5 or 10 for extension. Do not do Q2,4,6,7,8,9,11,12

In Leckie and Leckie - Exercise 2H Q1,3,4 and 5

Lesson 9 - Second derivatives of implicit functions

We can also use implicit differentiation to find higher order derivates for implicit functions

Example 1 find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $x^2 + y^2 = 25$ $\boldsymbol{\frac{dy}{dx} = -\frac{x}{y}}$

Use the quotient rule and implicit differentiation

$$\frac{d^2y}{dx^2} = -\left(\frac{y - x\frac{dy}{dx}}{y^2}\right)$$

Replace $\frac{dy}{dx}$ with $-\frac{x}{y}$

$$\frac{d^2y}{dx^2} = -\left(\frac{y - x(-\frac{x}{y})}{y^2}\right)$$

simplify

$$\frac{d^2y}{dx^2} = -\left(\frac{y + \frac{x^2}{y}}{y^2}\right) = -\left(\frac{\frac{y^2}{y} + \frac{x^2}{y}}{y^2}\right)$$

$$\frac{d^2y}{dx^2} = -\boldsymbol{\frac{y^2 + x^2}{y^3}}$$

Example 2 find the first and second derivative for $y^2 + xy = 2$

first derivative

$$2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$$

$$(2y + x)\frac{dy}{dx} = -y$$

$$\boldsymbol{\frac{dy}{dx} = -\frac{y}{2y + x}}$$

second derivative

$$-\left[\frac{(1)\frac{dy}{dx}(2y + x) - y(2\frac{dy}{dx} + 1)}{(2y + x)^2}\right]$$

$$-\left[\frac{(2y + x)\frac{dy}{dx} - 2y\frac{dy}{dx} - y}{(2y + x)^2}\right]$$

$$-\left[\frac{x\frac{dy}{dx} - y}{(2y + x)^2}\right] = \frac{y - x\frac{dy}{dx}}{(2y + x)^2}$$

$$\frac{y - x\left(\frac{-y}{2y + x}\right)}{(2y + x)^2} = \frac{y\left(\frac{2y + x}{2y + x}\right) - x\left(\frac{-y}{2y + x}\right)}{(2y + x)^2}$$

$$\frac{\frac{2y^2 + xy}{2y + x} + \frac{xy}{2y + x}}{(2y + x)^2} = \frac{2y^2 + xy + xy}{(2y + x)^3}$$

$$\boldsymbol{\frac{d^2y}{dx^2} = \frac{2y^2 + 2xy}{(2y + x)^3}}$$

Note:

- Use common factors of $\frac{dy}{dx}$ to simplify the second derivative before substitution
- Remember that $\frac{x}{y} + 1 = \frac{x}{y} + \frac{y}{y} = \frac{x+y}{y}$ and that $\frac{\frac{x}{y}+1}{y} = \frac{\frac{x+y}{y}}{y} = \frac{x+y}{y^2}$
- You can check your answers using a second implicit derivative calculator

***In your MIA textbook* – Try Exercise 6.5 Be careful - try Q1 and maybe one of 6 or 9.**
Do not do Q2,3,4 etc

In Leckie and Leckie - Exercise 2I page 68 Q1 and 2

Lesson 10 - Logarithmic differentiation

Here we use natural logs and the laws of logs to simplify functions before differentiation. This is particularly useful when the functions contain powers, roots and multiple factors

Example 1 Differentiate $f(x) = \ln(3x^4+7)^5$

Using laws of logs $f(x) = 5\ln(3x^4+7)$

Using the chain rule where $f(u) = 5\ln u$ and $u = 3x^4+7$

$$f'(x) = \frac{5}{u} \times 12x^3 = \boldsymbol{\frac{60x^3}{3x^4+7}}$$

Example 2 Differentiate $y = \ln\left(\frac{1+3x}{1-2x}\right)$

Using laws of logs $y = \ln(1+3x) - \ln(1-2x)$

$$\frac{dy}{dx} = \frac{1}{1+3x} \times 3 - \frac{1}{1-2x} \times -2 = \frac{3}{1+3x} + \frac{2}{1-2x}$$

This can then be expressed as a single fraction

$$\frac{dy}{dx} = \frac{3(1-2x)+2(1+3x)}{(1+3x)(1-2x)} = \frac{5}{1+x-6x^2}$$

Example 3 Differentiate $y = x^{\sin x}$ where $\sin x$ is a power.

Take natural logs of both sides $\qquad \ln y = \ln x^{\sin x}$

Use laws of logs $\qquad \ln y = \sin x \ln x$

Use implicit differentiation and the product rule $\qquad \frac{1}{y}\frac{dy}{dx} = \cos x \ln x + \frac{\sin x}{x}$

Simplify the RHS $\qquad \frac{1}{y}\frac{dy}{dx} = \frac{x\cos x \ln x + \sin x}{x}$

Multiply through by y $\qquad \frac{dy}{dx} = y \times \left(\frac{x\cos x \ln x + \sin x}{x}\right)$

Replace y with $x^{\sin x}$ $\qquad \frac{dy}{dx} = x^{\sin x}\left(\frac{x\cos x \ln x + \sin x}{x}\right)$

Logarithmic differentiation is used to ease the process of differentiation is found in the 2011 Past Paper

2011 Q7 A curve is defined by the equation $y = \frac{e^{\sin x}(2+x)^3}{\sqrt{1-x}}$

Calculate the gradient of this curve when $x = 0$

You can use a combination of the chain rule, product rule and quotient rule or

Take natural logs of both sides

$$\ln y = \ln\left(\frac{e^{\sin x}(2+x)^3}{\sqrt{1-x}}\right)$$

Use laws of logs

$$\ln y = \ln e^{\sin x} + \ln(2+x)^3 - \ln(1-x)^{\frac{1}{2}}$$

Remember that $\ln e^b = b$

$$\ln y = \sin x + 3\ln(2+x) - \frac{1}{2}\ln(1-x)$$

Use implicit differentiation

$$\frac{1}{y}\frac{dy}{dx} = \cos x + \frac{3}{2+x} - \frac{1}{2}\cdot\frac{(-1)}{1-x}$$

Simplify the RHS

$$\frac{1}{y}\frac{dy}{dx} = \cos x + \frac{3}{2+x} + \frac{1}{2(1-x)}$$

Multiply through by y

$$\frac{dy}{dx} = y\left(\cos x + \frac{3}{2+x} + \frac{1}{2(1-x)}\right)$$

From the original equation

When $x = 0$, $y = \frac{e^{\sin 0}(2+0)^3}{\sqrt{1-0}} = 8$

When $x = 0$ the gradient of the curve is:

$$\frac{dy}{dx} = 8\left(\cos 0 + \frac{3}{2+0} + \frac{1}{2(1-0)}\right)$$
$$= 8\left(1 + \frac{3}{2} + \frac{1}{2}\right) = \mathbf{24}$$

When the function reaches the stage of $\ln y$, the differentiation is eased, as is the resulting substitution into the derivative.

In your MIA textbook – In your textbook - Exercise 6.6 - Q 1 and 2 give the basics, Q5,6 and 7 are extension. Avoid Q3,4,8 – 10 far too much unnecessary algebra

In Leckie and Leckie - Exercise 2J page 69 Q1 to 4

Lesson 11 – Parametric equations

A parametric equation is where the x and y coordinates are both written in terms of another letter. This letter or **parameter** is usually either t or θ (when the parameter is an angle)

Parametric equations are most commonly used with circles and ellipses:

<u>Circles</u>

A circle with a centre at the origin and a radius r has the equation $x^2 + y^2 = r^2$ and this is an implicit function.

We can rearrange this in terms of x or y where $x = \pm\sqrt{r^2 - y^2}$ or $y = \pm\sqrt{r^2 - x^2}$

but each version is really two functions and each function only describes part of the circle.

So instead we describe this circle with two connected (**parametric**) equations which define the circle explicitly as x and y in terms of a third variable θ (the **parameter**)

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

These parametric equations can be returned to the form $x^2 + y^2 = r^2$ by eliminating θ

First square both equations $x^2 = r^2\cos^2\theta$ and $y^2 = r^2\sin^2\theta$

Add both equations together $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta)$

Use a trig identity $x^2 + y^2 = r^2$

So both $x^2 + y^2 = r^2$ and $x = r\cos\theta$, $y = r\sin\theta$ can be used to represent a circle with a centre at the origin and a radius of r

$x^2 + y^2 = r^2$ is called the **constraint equation**

$x = r\cos\theta$, $y = r\sin\theta$ is the **freedom equation** and θ is the **parameter.**

When given a parametric equation, you can always eliminate the parameter to get the cartesian (constraint) equation

Example 1 $\boldsymbol{x = at^2, \ y = 2at}$

$t = \sqrt{\frac{x}{a}}$ so $y = 2a\sqrt{\frac{x}{a}}$ $\boldsymbol{y^2 = 4ax}$

Example 2 For the parametric equation $x = 1 - 2\cos\theta, \quad y = 2 + 3\sin\theta$

Rearrange $x - 1 = -2\cos\theta, \quad y - 2 = 3\sin\theta$

Square both equations $(x-1)^2 = 4\cos^2\theta \quad (y-2)^2 = 9\sin^2\theta,$

Rearrange $\frac{(x-1)^2}{4} = \cos^2\theta, \quad \frac{(y-2)^2}{9} = \sin^2\theta,$

Add both equations together $$\frac{(x-1)^2}{4} + \frac{(y-2)^2}{9} = \cos^2\theta + \sin^2\theta,$$

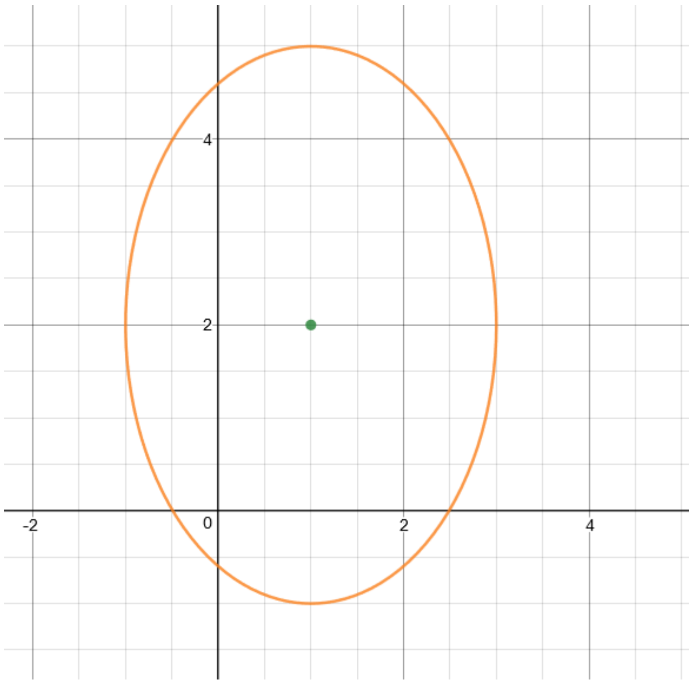
Use a trig identity $$\frac{(x-1)^2}{4} + \frac{(y-2)^2}{9} = 1$$

This parametric equation sketches an ellipse with:

a centre at (1,2)

a height of 6 units

and a width of 4 units



When graphing parametric curves with desmos use $(1 - 2\cos t, 2 + 3\sin t)$ and change the domain to $-\pi \leq t \leq \pi$

In your MIA textbook – Exercise 6.7, do Q1 and be very careful with this exercise.

In Leckie and Leckie - Exercise 2K Q1,2,3 and 5

Lesson 12 – Derivatives of Parametric equations

Given a pair of parametric equations $x = x(t)$ and $y = y(t)$

Use the chain rule to calculate the first derivative $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ or $\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

To calculate the second derivative, use the chain rule twice

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) \times \frac{dt}{dx}$$

Find the derivative with respect to t of the first derivative and then divide by $\frac{dx}{dt}$

Example 1 Find the first and second derivatives for the parametric equations

$x = 4 + 4t$ and $y = 3 - 3t^2$

$$\frac{dx}{dt} = 4, \quad \frac{dy}{dt} = -6t$$

Hence $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -6t \times \frac{1}{4} = -\frac{3t}{2}$

And $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) \times \frac{dt}{dx} = -\frac{3}{2} \times \frac{1}{4} = -\frac{3}{8}$

Example 2 A curve is defined by the equations $x = 5\cos\theta$ and $y = 5\sin\theta$

Find the equation of the tangent to the curve at the point where $\theta = \frac{\pi}{4}$

$\frac{dx}{d\theta}$ = -5 $\sin\theta$, $\frac{dy}{d\theta}$ = 5 $\cos\theta$, hence $\frac{dy}{dx} = -\frac{5\cos\theta}{5\sin\theta} = -\cot\theta.$

Where $\theta = \frac{\pi}{4}$ the point is $\left(\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right)$,

the gradient is $-\cot\left(-\frac{\pi}{4}\right) = -\frac{1}{\tan\left(\frac{\pi}{4}\right)} = -1$

Thus the equation of the tangent is $y = -x + 5\sqrt{2}$

Example 3 Given that $x = \sqrt{t}$ and $y = t^3 - t^2$ for $t > 0$,

use parametric differentiation to express $\frac{dy}{dx}$ in terms of t in simplified form and show that the second derivative takes the form $\frac{d^2y}{dx^2} = at^2 + bt$

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}}, \qquad \frac{dy}{dt} = 3t^2 - 2t$$

$$\frac{dy}{dx} = \frac{3t^2 - 2t}{\frac{1}{2\sqrt{t}}} = 2\sqrt{t}(3t^2 - 2t) = 6t^{\frac{5}{2}} - 4t^{\frac{3}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} \qquad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(6t^{\frac{5}{2}} - 4t^{\frac{3}{2}}\right) \times 2\sqrt{t}$$

$$= \left(15t^{\frac{3}{2}} - 6t^{\frac{1}{2}}\right) \times 2t^{\frac{1}{2}}$$

$$= 30t^2 - 12t$$

Example 4 With respect to a suitable coordinate system a particle has a position given by $(5t, 3t^2 - 2)$. Find the speed of the particle when $t = 2$ sec

The speed of a particle which is described by a parametric equation is given in terms of the time derivates of the x and y coordinates.

$$\text{Speed is } \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

As $\frac{dx}{dt} = 5$ $\frac{dy}{dt} = 6t$ and $t = 2$, then the speed is $\sqrt{(5)^2 + (6 \times 2)^2} = 13$

This is part of Calculus in context from Unit 2 and is only included here as the question is in the Unit Assessment for differentiation

In your MIA textbook – In your textbook – Exercise 6.8

- ***Q1 and 4 are straightforward skills-based questions***
- ***Q 2,3,5,6,7 etc use turning points (where $\frac{dy}{dx} = 0$) and concavity (taught later in the course so omit these parts just now)***
- ***Q8 to 10 would be extension***

In Leckie and Leckie - Exercise 2K Q1,2,3 and 5. For extension try Ex 2L Q5 and 6