1.1 Applying Algebraic skills to Matrices and Systems of Equations

Learning basic operations with matrices

- Add and subtract matrices
- Multiply by a scalar
- Find the transpose of a matrix
- Know properties of matrix algebra

A matrix is an array of numbers, essentially a compact table.

Example
The following tickets for the pantomime are sold:
Stalls – 10 adult and 27 child
Circle – 15 adult and 13 child
Balcony – 6 adult and 24 child

$$\begin{pmatrix} 10 & 27 \\ 15 & 13 \\ 6 & 24 \end{pmatrix}$$

The *order* of this matrix is its dimensions in terms of rows and columns *i.e.* 3 x 2.

Matrices can be added and subtracted if their orders are the same.

Example
These tickets are also sold:
Stalls – 15 adult and 24 child
Circle – 15 adult and 15 child
Balcony – 20 adult and 10 child

Find the total of each type of ticket.

$$\begin{pmatrix} 10 & 27 \\ 15 & 13 \\ 6 & 24 \end{pmatrix} + \begin{pmatrix} 15 & 24 \\ 15 & 15 \\ 20 & 10 \end{pmatrix} = \begin{pmatrix} 25 & 51 \\ 30 & 28 \\ 26 & 34 \end{pmatrix}$$

A matrix can be multiplied by a scalar by considering repeated addition.

Example

$$3\begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 3 \\ -9 & 6 \end{pmatrix}$$

The *transpose* of a matrix interchanges the rows and columns.

Example

$\begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 2 \end{pmatrix}'$ or sometimes $\begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 2 \end{pmatrix}^T = \begin{pmatrix} 3 & 4 \\ 2 & 5 \\ 1 & 2 \end{pmatrix}$

Ex 13.1 and Ex 13.2

Algebraic Properties of Matrices

The commutative law holds under addition *i.e.* $A + B = B + A$

The associative law holds *i.e.* $(A + B) + C = A + (B + C)$

Scalar multiplication is distributive over addition *i.e.* $k(A + B) = kA + kB$

$(A^T)^T = A$

$(A + B)^T = A^T + B^T$

$(kA)^T = kA^T$

Learning to multiply matrices

- Recognise when matrices can be multiplied
- Mutliply appropriately shaped matrices
- Understand what is meant by an identity and recognise the identity matrix (for multiplication)
- Know algebraic properties of matrices

To multiply matrices, it is necessary that the **number of columns** in the first matrix **equals** the **number of rows** in the second.
The position of an element in the product gives what should be multiplied to get it:

$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ *e.g.* a_{21} is obtained by multiplying row 2 (in the first matrix) by column 1 (in the second matrix).

Multiplying a 3×2 matrix and a 2×3 matrix:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} \end{pmatrix}$$

The product is a 3×3 matrix.

Example

$$\begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1\times2 + 1\times1 & 1\times-1 + 1\times2 & 1\times1 + 1\times0 \\ 2\times2 + 0\times1 & 2\times-1 + 0\times2 & 2\times1 + 0\times0 \\ 3\times2 + 5\times1 & 3\times-1 + 5\times2 & 3\times1 + 5\times0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & 1 \\ 4 & -2 & 2 \\ 11 & 7 & 3 \end{pmatrix}$$

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Identities

An identity has *no effect* under a given operation.

In ordinary algebra, under addition the identity is zero. $a + 0 = a$

Under multiplication the identity is 1. $a \times 1 = a$

In ,matrix algebra, under addition the identity is the zero matrix *e.g.*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Under multiplication the **identity matrix, I**, has ones on the leading diagonal and zeros elsewhere. *e.g.*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the 3×3 identity matrix.

Algebraic Properties of Matrices

In general, the commutative law **does not** hold for matrix multiplication $AB \neq BA$

$(AB)' = B'A'$ The transpose reversal rule.

The associative law hold for matrix multiplication, $A(BC) = (AB)C = ABC$.

The distributive law holds. $A(B + C) = AB + AC$

Learning to find the inverse of a matrix

- Know that the inverse of an operation produces the identity
- Know how to find the inverse of a 2x2 matrix
- Know how to find the determinant of a 2x2 matrix
- Use the determinant to identify singular matrices

Inverses

An inverse produces the identity under a given operation.

Under addition of real numbers the inverse of a is $-a$ because $a + (-a) = 0$

Under multiplication of real numbers the inverse of a is $\frac{1}{a}$ because $a \times \frac{1}{a} = 1$

For 2x2 matrices it can be shown that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

A x A^{-1} = I

$$A^{-1} = \begin{pmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The denominator of the common factor is called the **determinant, detA** or **|A|**

i.e. $\boxed{|A| = ad - bc}$ and $\boxed{A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$

- If det A = 0 then the inverse is undefined.
- A matrix with no inverse is called **singular**.
- If det A $\neq$ 0 the matrix is non-singular and invertible.

Examples

1. Find the determinant and inverse of the matrix $\begin{pmatrix} 3 & 5 \\ 2 & 7 \end{pmatrix}$.

$A = \begin{pmatrix} 3 & 5 \\ 2 & 7 \end{pmatrix}$ det $A = 3 \times 7 - 5 \times 2$

$= 11$

$A^{-1} = \frac{1}{11} \begin{pmatrix} 7 & -5 \\ -2 & 3 \end{pmatrix}$

2. By considering a matrix equation of the form $AX = B$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, prove that the system of equations $x + 4y = 6$
$2x + 8y = 10$ has no solutions.

Let $\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \end{pmatrix}$

$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$X = A^{-1}B$$

$\det A = 1 \times 8 - 4 \times 2 = 0$

Matrix A is <u>singular</u>. A^{-1} is undefined so X is undefined

i.e. the system of equations has no solutions.

Ex 13.6 Q1, 3, 4.
Ex 13.7 Q1 – 12.

Learning to find the determinant of a 3×3 *matrix*

- Know the meaning of the minor matrix of an element
- Know the pattern of signs for expression for the determinant
- Express the determinant of a 3×3 matrix as a combination of determinants of minor matrices.

In matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ the **minor matrix** of element a is $\begin{pmatrix} e & f \\ h & i \end{pmatrix}$. This is the matrix remaining when the row and the column containing the element a is deleted.

The determinant of a 3×3 matrix can be evaluated by combining the elements of one row multiplied by the determinant of their minor matrices respectively.

e.g.

$$\boxed{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}}$$

There are two other equivalent expressions using rows 2 and 3. The sign depends on the position of the element in the original matrix:

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

The expression using row 1 is often convenient *e.g.* application of 3×3 determinant to finding the vector product, but row 2 or 3 might save computation if either contains zero elements.

Example

Find $\begin{vmatrix} -1 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

$$\begin{vmatrix} -1 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -1\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} - 0 + 1\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= -1(-2-1) - 0 + 1(3-2)$$

$$= 3 + 1$$

$$= 4$$

Further Properties

$(AB)^{-1} = B^{-1}A^{-1}$

$|AB| = |A||B|$

$(A^{-1})^T = (A^T)^{-1}$

Learning to use Gaussian Elimination to solve systems of equations

- Recognise when a system of equations has a unique solution
- Recognise redundancy and write an expression for the infinite family of solutions
- Recognise inconsistency
- Express a system of equations as an augmented matrix
- Use elementary row operations to produce an upper triangular matrix and hence solve a system of equations

Attempt to solve:

1. $2x+3y=21 \xrightarrow{\times 3} 6x+9y=63$
   $3x+2y=19 \xrightarrow{\times 2} 6x+4y=38$
   $5y=25$
   $y=5$
   $x=3$

2. $3x+3y=6 \xrightarrow{\times 4} 12x+12y=24$
   $4x+4y=8 \xrightarrow{\times 3} 12x+12y=24$
   $0=0$

   Equation 2 is redundant
   The system of equations has infinitely many solutions of the form $(x, 2-x)$

3. $x+4y=6 \xrightarrow{\times 2} 2x+8y=12$
   $2x+8y=10 \xrightarrow{\times 1} 2x+8y=10$
   $0=2$

   The equations are inconsistent and have no solutions.

p260 Q1

Algebraic Equations	**Matrix Equation**	**Augmented Matrix**
$2x+3y=21$ $3x+2y=19$	$\begin{pmatrix}2 & 3\\3 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}21\\19\end{pmatrix}$	$\left(\begin{array}{cc:c}2 & 3 & 21\\3 & 2 & 19\end{array}\right)$

Elementary row operations include:

- switching order of rows
- multiplying a row by a constant
- adding or subtracting two rows

These are all equivalent to the standard operations used to solve simultaneous equations by elimination.

Solving the above example using elementary row operations:

$$\left(\begin{array}{cc|c} 2 & 3 & 21 \\ 3 & 2 & 19 \end{array}\right) \begin{array}{l} r_1 \to 3r_1 \\ r_2 \to 2r_2 \end{array}$$

$$\left(\begin{array}{cc|c} 6 & 9 & 63 \\ 6 & 4 & 38 \end{array}\right) r_2 \to r_1 - r_2$$

$$\left(\begin{array}{cc|c} 6 & 9 & 63 \\ 0 & 5 & 25 \end{array}\right)$$

r_2: $5y = 25$

$y = 5$

r_1: $6x + 9 \times 5 = 63$

$x = 3$

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Learning to solve a 3X3 *system of equations using Gaussian Elimination*

- Write down a augmented matrix
- Produce zeros at a_{21}, a_{31} and a_{32} <u>in that order</u>
- Write down simplified algebraic equations and solve for x, y and z.

Example

A parabola passes through the points (-1, 6), (1, 2) and (3, 22). Find the equation of the parabola.

Equation of parabola: $y = ax^2 + bx + c$

$6 = a - b + c$

$2 = a + b + c$

$22 = 9a + 3b + c$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 1 & 1 & 1 & 2 \\ 9 & 3 & 1 & 22 \end{array}\right) \begin{array}{l} \\ r_2 \to r_2 - r_1 \\ r_3 \to r_3 - 9r_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 2 & 0 & -4 \\ 0 & 12 & -8 & -32 \end{array}\right) \begin{array}{l} \\ \\ r_3 \to r_3 - 6r_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 2 & 0 & -4 \\ 0 & 0 & -8 & -8 \end{array}\right)$$

$\boxed{\text{p265 Ex14.4}}$

r_3: $-8c = -8$

$c = 1$

r_2: $2b = -4$

$b = -2$

r_1: $a - b + c = 6$

$a + 2 + 1 = 6$

$a = 3$

The equation is $y = 3x^2 - 2x + 1$

Learning to recognise redundancy and inconsistency in a 3x3 system of equations.

In each case use elementary row operations on the augmented matrix to produce an upper triangular matrix:

(a)
$$\begin{aligned} x+2y+2z&=11\\ x-y+3z&=8\\ 4x-y+11z&=35 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11\\ 1 & -1 & 3 & 8\\ 4 & -1 & 11 & 35 \end{array}\right)\begin{array}{l} \\ r_2-r_1\\ r_3-4r_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11\\ 0 & -3 & 1 & -3\\ 0 & -9 & 3 & -9 \end{array}\right)\begin{array}{l} \\ \\ r_3-3r_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11\\ 0 & -3 & 1 & -3\\ 0 & 0 & 0 & 0 \end{array}\right)$$

(b)
$$\begin{aligned} x+2y+2z&=11\\ 2x-y+z&=8\\ 3x+y+3z&=18 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11\\ 2 & -1 & 1 & 8\\ 3 & 1 & 3 & 18 \end{array}\right)\begin{array}{l} \\ r_2-2r_1\\ r_3-3r_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11\\ 0 & -5 & -3 & -14\\ 0 & -5 & -3 & -15 \end{array}\right)\begin{array}{l} \\ \\ r_3-r_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11\\ 0 & -5 & -3 & -14\\ 0 & 0 & 0 & -1 \end{array}\right)$$

In (a) the row of zeros tells us that the one equation is redundant. There is no unique solution to the system but an infinite family of solutions can be obtained by writing x and y in terms of z.

r_2: $-3y+z=-3$

$$y=\frac{z+3}{3}$$

r_1: $x+2y+2z=11$

$$x+\frac{2z+6}{3}+2z=11$$

$$x=\frac{27-8z}{3}$$

In (b) row 3 suggest that $0z=-1$. This tells us that the equations are inconsistent and hence there are no solutions.

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Learning the effect of experimental data on systems of equations

- Calculate errors and percentage errors
- Recognise a matrix which is ill-conditioned
- Understand the implication of an ill-conditioned matrix.

Use Gaussian Elimination to solve:

$$11x + 12y + 3z = 44$$
$$10x + 10y + z = 33$$
$$42x + 43y + 6z = 146$$

$$\left(\begin{array}{ccc|c} 11 & 12 & 3 & 44 \\ 10 & 10 & 1 & 33 \\ 42 & 43 & 6 & 146 \end{array}\right) \quad \left(\begin{array}{ccc|c} 462 & 504 & 126 & 1848 \\ 0 & 10 & 19 & 77 \\ 462 & 473 & 66 & 1606 \end{array}\right) \quad \left(\begin{array}{ccc|c} 11 & 12 & 3 & 44 \\ 0 & 10 & 19 & 77 \\ 0 & 0 & 11 & 33 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 110 & 120 & 30 & 440 \\ 110 & 110 & 11 & 363 \\ 42 & 43 & 6 & 146 \end{array}\right) \quad \left(\begin{array}{ccc|c} 11 & 12 & 3 & 44 \\ 0 & 10 & 19 & 77 \\ 0 & 31 & 60 & 242 \end{array}\right)$$

$z = 3$

$10y + 19z = 77$

$10y = 20$

$y = 2$

$$\left(\begin{array}{ccc|c} 110 & 120 & 30 & 440 \\ 0 & 10 & 19 & 77 \\ 42 & 43 & 6 & 146 \end{array}\right) \quad \left(\begin{array}{cccc} 11 & 12 & 3 & 44 \\ 0 & 310 & 589 & 2387 \\ 0 & 310 & 600 & 2420 \end{array}\right)$$

$11x + 24 + 9 = 44$

$11x = 11$

$x = 1$

Suppose $x = 51$ and $y = 50$ but that these are measurements to the nearest whole number. What is the smallest and largest value of $x + y$ and of $x - y$? What is the percentage error in each case?

Investigate the implication of the "44" in the above system of equations being a measurement to the nearest whole number.

e.g.

$$\left(\begin{array}{ccc|c} 11 & 12 & 3 & 43.5 \\ 10 & 10 & 1 & 33 \\ 42 & 43 & 6 & 146 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 11 & 12 & 3 & 44.5 \\ 10 & 10 & 1 & 33 \\ 42 & 43 & 6 & 146 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 11 & 12 & 3 & 43.5 \\ 0 & 10 & 19 & 72 \\ 0 & 0 & 11 & 22 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 11 & 12 & 3 & 44.5 \\ 0 & 10 & 19 & 82 \\ 0 & 0 & 11 & 88 \end{array}\right)$$

$z = 2$

$y = 3.4$

$x = -0.3$

$z = 8$

$y = -7$

$x = 9.5$

When a small change in any one value leads to a disproportionate change in the solution the matrix is said to be ill-conditioned. If a system is ill-conditioned we can have no confidence in the solution obtained.
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Learning to find the inverse of a matrix using elementary row operations

Steps:

- Write the identity matrix beside the matrix to be inverted.
- Perform the same EROs on both matrices until the original becomes the identity.
- The matrix produced is the inverse.

Example

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right) \; r_3 - 3r_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -3 & -2 & -3 & 0 & 1 \end{array}\right) \; r_3 + 3r_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 3 & 1 \end{array}\right) \; \begin{array}{l} r_1 - r_3 \\ r_2 - r_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & -3 & -1 \\ 0 & 1 & 0 & 3 & -2 & -1 \\ 0 & 0 & 1 & -3 & 3 & 1 \end{array}\right) \; r_1 - 2r_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 1 \\ 0 & 1 & 0 & 3 & -2 & -1 \\ 0 & 0 & 1 & -3 & 3 & 1 \end{array}\right)$$

The inverse is:

$$\begin{pmatrix} -2 & 1 & 1 \\ 3 & -2 & -1 \\ -3 & 3 & 1 \end{pmatrix}$$

Check:

$$\begin{pmatrix} -2 & 1 & 1 \\ 3 & -2 & -1 \\ -3 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 3 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Learning to use and find transformation matrices

- Find the image of a point by multiplying by the transformation matrix
- Construct a transformation matrix for a simple linear transformation

A linear transformation is one which can be described:

$(x, y) \to (ax + by, cx + dy)$ with $a, b, c, d \in \boldsymbol{R}$

If point $P\ (x, y)$ has image $P'(x', y')$ then

$$\begin{matrix} x' = ax + by \\ y' = cx + dy \end{matrix} \Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

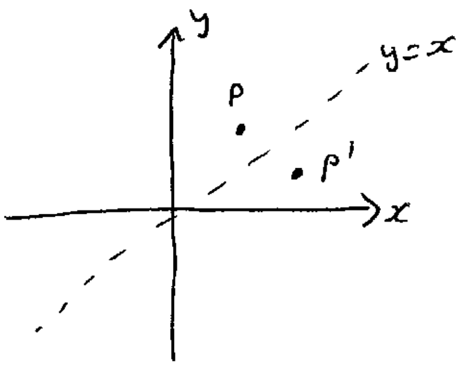
Thus the position vector of P' is obtained by pre-multiplying the position vector of P by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Example

Find the image of $P = (2, 3)$ under the transformation $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$$P' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$P' = (3, 2)$



The transformation is a reflection in the line $y = x$

Contructing the transformation matrix

Consider the points (1, 0) and (0, 1).

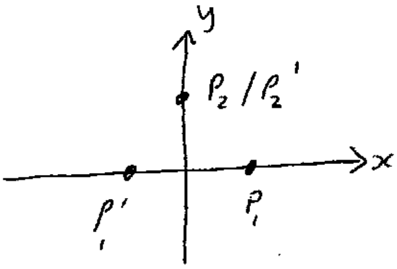
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

Thus the image of (1, 0) is (a, c) and the image of (0, 1) is (b, d)

Example

Find the transformation matrix for a reflection in the y-axis.



$$\overset{a\ \ c}{(1,0) \to (-1,0)} \quad \overset{b\ \ d}{(0,1) \to (0,1)}$$

The matrix is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

- A transformation can be reversed by pre-multiplying the image by the inverse of the transformation matrix.
- If a point is its own image under a particular transformation then it is said to be **invariant**.

Example

Find the invariant points under the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$-x = x$$

$$2x = 0$$

$$x = 0$$

All points on the line $x = 0$ (y-axis) are invariant under the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$