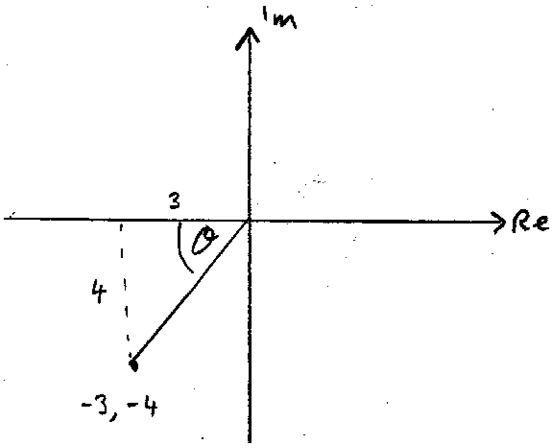
1.3 Applying Geometric skills to Complex Numbers

Learning the properties of the modulus and argument of a complex number

- Know how to multiply complex numbers in polar form
- Know how to divide complex numbers in polar form
- Know De Moivre's Theorem

Plot $Z_1 = -3 - 4i$ on an argand diagram and find $|Z_1|$ and two different expressions for arg Z_1.



$|z_1| = \sqrt{3^2+4^2} = 5$

$\theta = \tan^{-1}\frac{4}{3} \approx 53°$

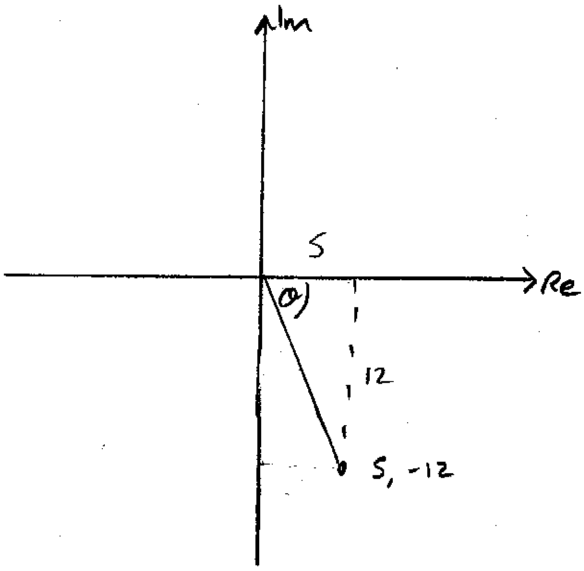
$\arg z_1 \approx 180 + 53 = 233°$

or

$\arg z \approx -127°$

Which is the principal argument? The principal argument is $-127°$

Plot $Z_2 = 5 - 12i$ and find $|Z_2|$ and arg Z_2.



$|Z_2| = \sqrt{5^2+12^2} = 13$

$\theta = \tan^{-1}\left(\frac{12}{5}\right) \approx 67°$

$\arg Z_2 = -67°$

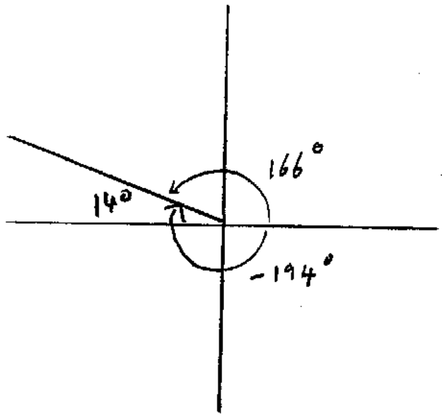
Find $|Z_1Z_2|$ and arg (Z_1Z_2).
Conjecture a relationship between $|Z_1|, |Z_2|$ *and* $|Z_1Z_2|$ and between arg Z_1, arg Z_2 and arg(Z_1Z_2).

$$Z_1Z_2 = (-3-4i)(5-12i) = -63+16i$$

$$|Z_1Z_2| = \sqrt{63^2+16^2} = 65$$

$$\theta = \tan^{-1}\left(\frac{16}{63}\right) = 14^\circ$$

$$\arg(Z_1Z_2) = 180-14 = 166^\circ$$



Conjecture

$|Z_1Z_2| = |Z_1| \times |Z_2|$ since $5 \times 13 = 65$

$\arg(Z_1Z_2) = \arg Z_1 + \arg Z_2$ since $-127 + -67 = -194^\circ$ which is equivalent to 166°

Proof

Let $Z_1 = r_1(\cos A + i\sin A)$ and $Z_2 = r_2(\cos B + i\sin B)$

$$Z_1Z_2 = r_1r_2(\cos A + i\sin A)(\cos B + i\sin B)$$
$$= r_1r_2(\cos A\cos B + i\sin B\cos A + i\sin A\cos B + i^2\sin A\sin B)$$
$$= r_1r_2(\cos A\cos B - \sin A\sin B + i(\sin A\cos B + \sin B\cos A))$$
$$= r_1r_2(\cos(A+B) + i\sin(A+B))$$

This result extends to division:
$\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|}$ and arg (Z_1Z_2) = arg Z_1 – arg Z_2

And to powers:
$|Z^2| = |Z|^2$ *and* $\arg(Z^2) = 2\arg Z$.

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- Add or subtract 2π to ensure that arguments are in the range $-\pi$ to π
- Use higher results to simplify
- Consider what will happen with higher powers of Z

De Moivre's Theorem

$|Z^2| = |Z|^2$ *and* $\arg(Z^2) = 2\arg Z$
thus if $Z = r(\cos\theta + i\sin\theta)$ then $Z^2 = r^2(\cos 2\theta + i\sin 2\theta)$.

In general $Z^n = (r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

Use proof by induction to show that De Moivre's theorem hold for all positive integers, n.

Let $n=1$ $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$ True $n=1$

Assume true $n=k$ $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

Consider $n=k+1$ $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$

$$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$

$$= \cos k\theta\cos\theta + i\sin\theta\cos k\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta$$

$$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\sin k\theta\cos\theta + \sin\theta\cos k\theta)$$

$$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$$

$$= \cos(k+1)\theta + i\sin(k+1)\theta$$

True $n=k+1$

The theorem is true for $n=1$ and if true for $n=k$, then true for $n=k+1$. Hence by induction, true $\forall n \in \mathbb{N}$.

De Moivre's Theorem in fact holds $\forall\, n \in \boldsymbol{R}$ but proof for non-integer values is beyond the scope of the course.

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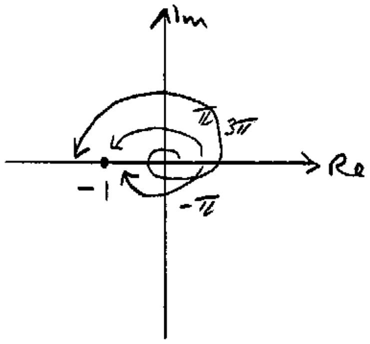
Learning to find the n^{th} roots of a complex number

- Plot on an argand diagram and write number in polar form
- n equivalent expressions for the argument
- Apply De Moivre's Theorem
- Roots will be equally spaced around the argand diagram (radius = modulus)

Example
Find the cube roots of -1.

Let $Z^3 = -1$ so $Z = (-1)^{1/3}$

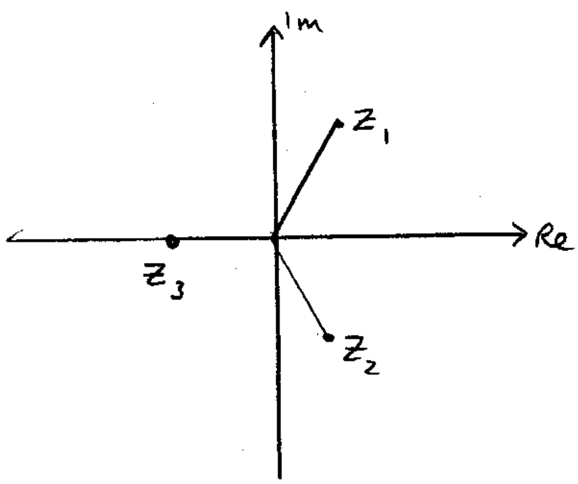


$|Z^3| = 1$ $\qquad$ $\arg Z^3 = \pi = -\pi = 3\pi$

$Z^3 = \cos\pi + i\sin\pi = \cos(-\pi) + i\sin(-\pi) = \cos 3\pi + i\sin 3\pi$

$Z = (\cos\pi + i\sin\pi)^{1/3} = (\cos(-\pi) + i\sin(-\pi))^{1/3} = (\cos 3\pi + i\sin 3\pi)^{1/3}$

$Z_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ $\quad$ $Z_2 = \cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3}$ $\quad$ $Z_3 = \cos\pi + i\sin\pi$



Learning to identify the locus of a point in the complex plane

- Draw a locus using geometrically
- Identify a locus using an algebraic approach

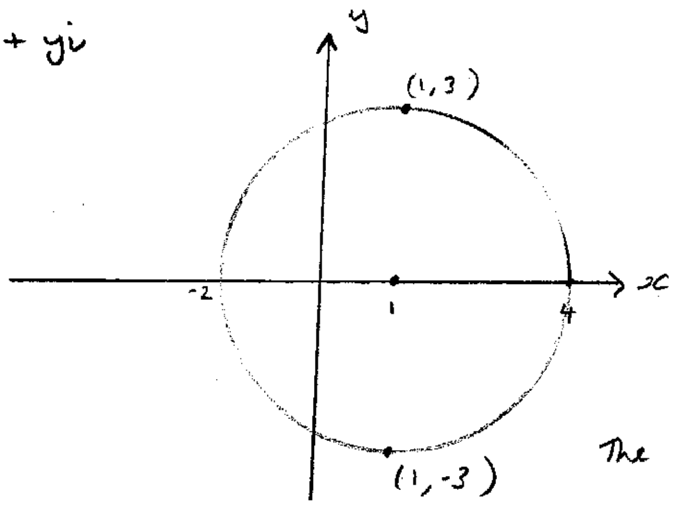
Example 1

Identify the locus in the complex plane given by

$$|Z-1|=3$$

$|Z-1|$ indicates the distance from the complex number Z to the number 1 and this has to be always equal to 3, *i.e.* we have a circle, centred on 1 with radius 3.

If $z = x + yi$

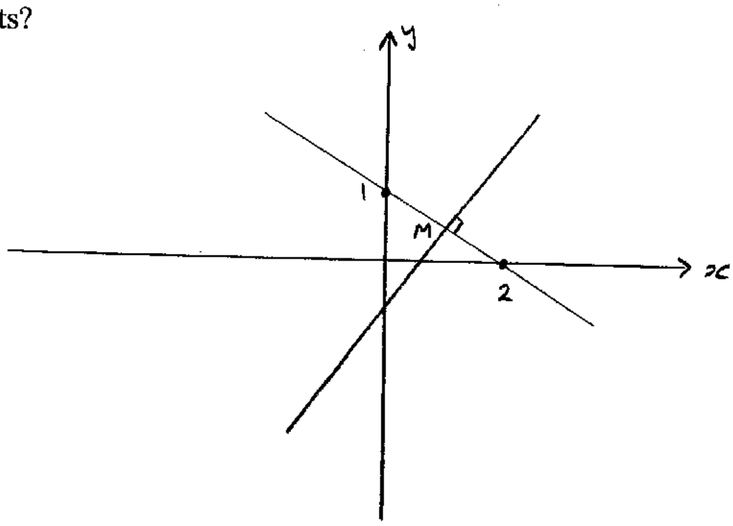


The locus is $(x-1)^2 + y^2 = 9$

Example 2

$|Z-2|=|Z-i|$

This time the distance from Z to 2 is the same as the distance from Z to *i*. Where are these points?



The required locus is the perpendicular bisector of the line joining (2,0) and (0,1)

$M = (1, \frac{1}{2})$

$m_1 = \frac{-1}{2} \qquad m_{\perp r} = 2$

The equation is

$y - \frac{1}{2} = 2(x-1)$

$2y - 1 = 4(x-1)$

$2y - 4x + 3 = 0$

Use higher work to determine the equation of the locus.

This can also be done by solving the given equation algebraically.

$|Z_1 - Z_2|$ is the distance between two points in the complex plane $Z_1 = (x_1, y_1)$ and $Z_2 = (x_2, y_2)$ so $|Z_1 - Z_2|$ can be found using the distance formula:

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$|Z - 2| = |Z - i|$

$\sqrt{(x-2)^2 + (y-0)^2} = \sqrt{(x-0)^2 + (y-1)^2}$

$(x-2)^2 + y^2 = x^2 + (y-1)^2$

$x^2 - 4x + 4 + y^2 = x^2 + y^2 - 2y + 1$

$-4x + 4 = -2y + 1$

$\underline{\underline{2y - 4x + 3 = 0}}$