	Prelim Revision 2 Non-Calculator	30
1	For $f(x) = \sin x \cos x$ find $f'(x)$.	2
2	Find $$\int \frac{6x^2 - 2}{x^3 - x + 3}\ dx$$	3
3	Write down and simplify the general term for $\left(3p^3 - \frac{2}{p}\right)^4$ Hence or otherwise find the term independent of p	5
4	Given that $z_1 = 1 + 2i$ and $z_2 = p - 4i$ Find $z_1\overline{z_2}$ and the value of p such that $z_1\overline{z_2}$ is a real number	3
5	Use Gaussian Elimination to give an expression for α in terms of λ $$x - y + 2z = -3$$ $$-x + 2y - 3z = 2$$ $$2x - y + \alpha z = 1$$ Explain what happens when $\alpha = 3$ Find the solution corresponding to $\alpha = -13$	6
6	Using the substitution $t = 1 + \tan x$, show that $$\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1 + \tan x} dx = \ln 2$$	5
7	Obtain the general solution of the differential equation $$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 3x + 10$$	6

	Prelim Revision 2 Calculator	50
8	A geometric sequence has third and fifth terms 24 and 6 respectively (a) Calculate the value of the common ratio (b) State why the associated geometric series has a sum to infinity (c) Find the value of this sum to infinity	 2 1 2
9	Matrices C and D are given by: $C = \begin{pmatrix} -2 & -1 & 1 \\ 3 & -1 & 1 \\ 2 & 0 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & -4 & 1 \\ k+5 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ (a) Find C^T the transpose of C (b) (i) find and simplify an expression for the determinant of D (ii) State the value of k such that D^{-1} does not exist	 1 2 1
10	(a) Obtain partial fractions for $\frac{x-4}{x^2+x-2}$ (b) Hence find $\int \frac{x-4}{x^2+x-2}\ dx$	3 3
11	Verify that $1-3i$ is a solution of $z^4-4z^3+11z^2-14z-30=0$ Hence find all the solutions to $z^4-4z^3+11z^2-14z-30=0$	 5
12	Use the division algorithm to express 1243_{10} in base 6	3

13	Given that $p(n) = n^2 - n$, where n is a positive integer, consider the statements **A** $p(n)$ is always even **B** $p(n)$ is a multiple of 3 For each statement, prove if it is true or, otherwise, disprove it	4
14	A curve is defined by the equation $xy^2 - 2x^2y = 3$ for $x > 0$ and $y > 0$ Use implicit differentiation to find the equation of the tangent to the curve when $x = 1$	5
15	Tickets for a concert go on sale 30 days before the concert. Over the month the rate at which the tickets sell can be modelled by $$\frac{dP}{dt} = \frac{t^2}{50}(50 - P)$$ Where the percentage of tickets sold at time t days after the sale begins is P%. Obviously when t = 0, $P = 0$ Express P in terms of t explicitly	6
16	A function is defined over a suitable domain by $f(x) = \frac{x^3}{x^2-4}$ (a) Prove that it is an odd function (b) Find the equations of the vertical asymptotes (c) Find the equation of the non-vertical asymptote (d) Calculate the x-coordinates of the stationary points and determine their nature (e) Sketch the curve $f(x) = \frac{x^3}{x^2-4}$	 2 1 2 4 3

	Prelim Revision 2 – Answers
1	$f'(x) = \cos x \cos x - \sin x \sin x \ = \cos^2 x - \sin^2 x \ \ or \ \cos 2x$
2.	This is a common form where $\int \frac{f'(x)}{f(x)}\ dx = \ln\lvert f(x)\rvert \ + C$ $\int \frac{6x^2-2}{x^3-x+3}\ dx = 2\int \frac{3x^2-1}{x^3-x+3}\ dx = 2\ln\lvert x^3-x+3\rvert + C$
3	The general term is $\binom{4}{r}(3p^3)^{4-r}\left(\frac{-2}{p}\right)^r = \binom{4}{r}(3^{4-r})(-2)^r(p^{12-3r})(p^{-r}) = \binom{4}{r}(3^{4-r})(-2)^r(p^{12-4r})$ The independent term happens when, $12-4r=0, r=3$ $\binom{4}{3}(3^1)(-2)^3(p^{12-12}) = 4\times 3\times -8 = \ -96$
4	$\overline{z_2} = p+4i,$ $z_1\overline{z_2} = (1+2i)(p+4i) = p+4i+2ip+8i^2 = p-8+(4+2p)i$ If $z_1\overline{z_2}$ is a real number $\quad 4+2p=0,\ p=\ -2$
5	$\begin{pmatrix} 1 & -1 & 2 : & -3 \\ -1 & 2 & -3: & 2 \\ 2 & -1 & \alpha : & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & -1 & 2 & : & -3 \\ 0 & 1 & -1 & : & -1 \\ 0 & 1 & \alpha-4: & & 7 \end{pmatrix} \ R_2+R_1,\ \ R_3-2R_1$ $\begin{pmatrix} 1 & -1 & 2 & : & -3 \\ 0 & 1 & -1 & : & -1 \\ 0 & 0 & \alpha-3: & & 8 \end{pmatrix} \quad R_3-R_2$ $(\alpha-3)z = 8, \qquad z = \frac{8}{\alpha-3}$ When $\alpha = 3$ the final row is inconsistent since $0 \neq 8$ so there are no solutions When $\alpha = \ -13$, $\qquad z = \frac{8}{-16} z = \ -\frac{1}{2},\ \ y-z=-1,\ \ y = \ -\frac{3}{2},$ $x + \frac{3}{2} - 1 = -3, \quad x = -\frac{7}{2}$

6	$u = 1 + \tan x\ ,\ \ du = \sec^2 x\ dx \qquad x = \frac{\pi}{4}\ \ u = 2,\ \ x = 0\ \ u = 1$ $\int_1^2 \frac{1}{u} du \ = [\ln u]_1^2 = \ln 2 - \ln 1 = \ln 2\ \ \text{as required}$
7	For the complimentary function $\qquad 4m^2 + 4m + 1 = 0,$ $(2m+1)^2 = 0\ \ m = -\frac{1}{2}$ $\boldsymbol{y = Ae^{-\frac{1}{2}x} + Bxe^{-\frac{1}{2}x}}$ For the particular integral $\qquad y = mx + c,\ \frac{dy}{dx} = m,\ \frac{d^2y}{dx^2} = 0$ $4 \times 0 + 4m + mx + c = 3x + 10$ $mx = 3x\ \ so\ m = 3, \quad 4m + c = 10\ \ so\ c = \ -2$ $\boldsymbol{y = 3x - 2}$ **Thus the general solution is** $\ \boldsymbol{y = \ Ae^{-\frac{1}{2}x} + Bxe^{-\frac{1}{2}x} + 3x - 2}$
8	$ar^2 = 24\ \text{and}\ ar^4 = 6,\ \ \frac{6}{24} = \frac{r^4}{r^2}\ \text{so}\ \frac{1}{4} = r^2\ \ \text{the common ratio is}\ \frac{1}{2}$ $\text{this series has a sum to infinity as}\ \left\lvert\frac{1}{2}\right\rvert < 1$ first term is 96 sun to infinity is $S_\infty = \frac{96}{1-\frac{1}{2}} = 192$
9	$C^T = \begin{pmatrix} -2 & 3 & 2 \\ -1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$ $\det D \ = 1 \times \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 4 \times \det\begin{pmatrix} k+5 & 1 \\ 2 & 0 \end{pmatrix} + 1 \times \det\begin{pmatrix} k+5 & 0 \\ 2 & 1 \end{pmatrix}$ $= \ -1 - 8 + k + 5 = k - 4$ For D^{-1} to not exist $k - 4 = 0,\ \ k = 4$

10

$$\frac{x-4}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1} \qquad A = 2, \qquad B = -1$$

$$\int \frac{x-4}{x^2+x-2}\ dx = 2\int \frac{1}{x+2}\ dx - \int \frac{1}{x-1}\ dx$$

$$= 2\ln(x+2) - \ln(x-1) + C \qquad = \ln\left(\frac{(x+2)^2}{x-1}\right) + C$$

11

$$\begin{array}{c|ccccc} 1-3i & 1 & -4 & 11 & -14 & -30 \\ & 0 & 1-3i & -12+6i & 17+9i & 30 \\ \hline & 1 & -3-3i & -1+6i & 3+9i & \boxed{0} \end{array}$$

No remainder so $i - 3i$ is a solution to this polynomial. As is $z = 1+3i$

$$\begin{array}{c|cccc} 1+3i & 1 & -3-3i & -1+6i & 3+9i \\ & 0 & 1+3i & -2-6i & -3-9i \\ \hline & 1 & -2 & -3 & \boxed{0} \end{array}$$

$$z^4 - 4z^3 + 11z^2 - 14z - 30 = (z-1-3i)(z-1+3i)(z^2-2z-3)$$
$$= (z-1-3i)(z-1+3i)(z-3)(z+1)$$

Thus the solutions are $z = 1 \pm 3i,\ z = 3, z = -1$

12

$1243 = 6 \times 207 + 1$

$207 = 6 \times 34 + 3$

$34 = 6 \times 5 + 4$

$5 = 0 \times 6 + 5$ $\qquad$ $\mathbf{1243_{10} = 5431_6}$

13

Positive integers take the form $2n$ or $2n+1$

For $2n$, $p(2n) = (2n)^2 - 2n = 4n^2 - 2n = 2(2n^2 - n)$ which is even for all positive values of n

For $2n+1$, $\quad p(2n+1) = (2n+1)^2 - (2n+1) = 4n^2 + 2n = 2(2n^2+n))$

which is even for all positive values of n

therefore **A** is true

For $n = 1,\ p(n) = 5^2 - 5 = 20$ this is not divisible by 3 statement **B** is false

14	When $x = 1$, $y^2 - 2y - 3 = 0$ $(y-3)(y+1) = 0$, $y > 0$ $so\ y = 3$ Implicit differentiation $$\frac{d}{du}(xy^2) + \frac{d}{du}(-2x^2y) = \frac{d}{du}(3)$$ $$y^2 + 2xy\frac{dy}{dx} - 4xy - 2x^2\frac{dy}{dx} = 0$$ $$(2xy - 2x^2)\frac{dy}{dx} = 4xy - y^2$$ $$\frac{dy}{dx} = \frac{4xy - y^2}{2xy - 2x^2}$$ For $x = 1, y = 3$ gradient is $\frac{12-9}{6-2} = \frac{3}{4}$ Equation of the straight line is $y = \frac{3}{4}x + \frac{9}{4}$ $or\ 4y = 3x + 9$
15	Separating the variables for $\frac{dP}{dt} = \frac{t^2}{50}(50 - P)$ $$\frac{1}{50-P}dP = \frac{1}{50}t^2dt$$ $$\int \frac{1}{50-P}dP = \int \frac{1}{50}t^2dt$$ $$-\ln(50-P) = \frac{1}{150}t^3 + C$$ $$\ln(50-P) = C - \frac{1}{150}t^3$$ when $t = 0$, $P = 0$ $\ln(50) = C$ $$\ln(50-P) = \ln(50) - \frac{1}{150}t^3$$ $$50 - P = e^{\ln(50) - \frac{1}{150}t^3}$$ $$50 - P = 50e^{-\frac{1}{150}t^3}$$ $$P = 50 - 50e^{-\frac{1}{150}t^3}$$

16

(a) $f(-x) = \frac{(-x)^3}{(-x)^2-4} = \frac{-x^3}{x^2-4} = -\left(\frac{-x^3}{x^2-4}\right) = -f(x)$ hence this is an odd function

(b) Vertical asymptotes are $x = \pm 2$

(c) By long division $f(x) = x + \frac{4x}{x^2-4}$, non-vertical asymptote is $y = x$

(d) $f'(x) = \frac{3x^2(x^2-4)-2x(x^3)}{(x^2-4)^2} = \frac{x^4-12x^2}{(x^2-4)^2} = \frac{x^2(x^2-12)}{(x^2-4)^2}$

$f'(x) = 0, \quad x^4 - 12x^2 = 0, \quad x^2(x^2-12) = 0, \quad x = 0 \;\; x = \pm\sqrt{12}$

$$f''(x) = \frac{(4x^3-24x)(x^2-4)^2 - 2(x^2-4)(2x)(x^4-12x^2)}{(x^2-4)^4} = \frac{8x^3+96x^2}{(x^2-4)^3} = \frac{8x(x^2+12)}{(x^2-4)^3}$$

$f''(x) = 0$, $x = 0$ possible point of inflexion at $(0,0)$

x	$\rightarrow$	0	$\rightarrow$
$f''(x)$	positive	zero	negative
Concavity	up		down

Change of concavity, hence $(0,0)$ is a point of inflexion

$f''(\sqrt{12}) > 0$ min stationary point when $x = \sqrt{12}$

$f''(-\sqrt{12}) < 0$ max stationary point when $x = -\sqrt{12}$