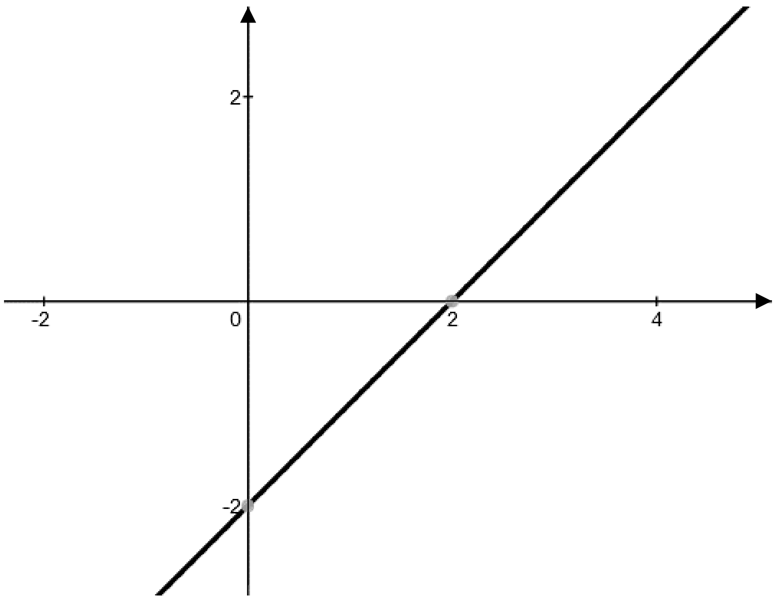
	Prelim Revision 3 Non-Calculator	30
1	Use binomial expansion to expand $\left(\frac{1}{2}x-3\right)^4$ and simplify your answer	3
2	Differentiate (a) $f(x)=e^x(5-x)^3$ (b) $g(x)=\sin^{-1}5x$	5
3	Calculate the inverse of the matrix $\begin{pmatrix} 5 & \alpha \\ -2 & 2 \end{pmatrix}$. For what value of α is this matrix singular	4
4	Verify that i is a solution of $z^4+4z^3+3z^2+4z+2=0$. Hence find all solutions	5
5	Below is a diagram showing the graph of a linear function $y=f(x)$  On separate diagrams show $y=\lvert f(x)\rvert$ and $y=\lvert f(x)+2\rvert$	3
6	A curve is defined parametrically for all t by the equations $x=\sin 2t,\quad y=\cos 2t$ Obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ as functions of t	5
7	Use integration by parts to find $\int x^2e^{-x}\,dx$	5

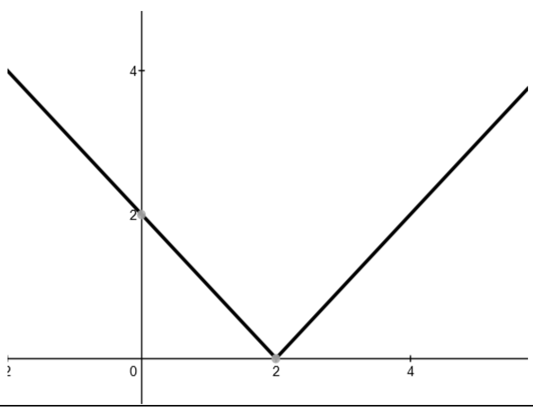
	Prelim Revision 3 Calculator	50
8	The first term of an arithmetic sequence is 5 and the twentieth term is 62 Obtain the sum of the first 40 terms	4
9	Identify the locus in the complex plane given by $\lvert z+i \rvert = 5$	3
10	Use Gaussian elimination to solve this system of equations $$\begin{aligned} a - b + c &= 1 \\ 2a + b - 3c &= -6 \\ a - b + 2c &= 2 \end{aligned}$$	5
11	Use the substitution $u = x^4$ to obtain $$\int \frac{2x^3}{1+x^8}\,dx$$	4
12	Use the Euclidean Algorithm to obtain the greatest common divisor of 221 and 2678 expressing it in the form $221a + 2678b$	5
13	A matrix A is such that $A^2 = 4A - 3I$ Find integers p and q such that $A^4 = pA + qI$	4
14	Prove by induction that, for all natural numbers n, $$\sum_{r=1}^{n} 3(r^2 - r) = (n-1)n(n+1)$$ Hence evaluate $$\sum_{r=11}^{40} 3(r^2 - r)$$	4 2
15	Obtain the general solution of the differential equation $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$$ Find the particular solution corresponding to the initial conditions $y = 2$ and $\frac{dy}{dx} = 5$ when $x = 0$	4 3

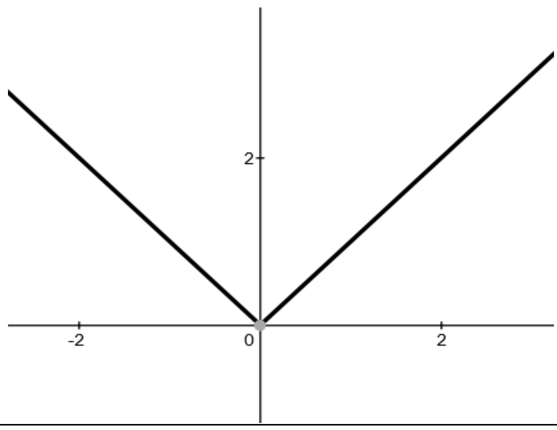
16	The function g is defined as $g(x) = \frac{x^2+5}{x+2}$, $x \neq -2$, $x \in \mathbb{R}$ (a) Obtain the equations of the asymptotes of g (b) Find the coordinates of the point where the graph of $y = g(x)$ crosses the y-axis (c) Find the coordinates and the nature of the stationary points of g (d) Sketch the graph of $y = g(x)$, indicating the features found in (a), (b) and (c)	3 1 5 3

<table>
<tr><td></td><td>Prelim Revision 3 – Answers</td></tr>
<tr><td>1</td><td>$$\left(\frac{1}{2}x\right)^4 + 4\left(\frac{1}{2}x\right)^3 \times (-3)^1 + 6\left(\frac{1}{2}x\right)^2 \times (-3)^2 + 4\left(\frac{1}{2}x\right) \times (-3)^3 + (-3)^4$$
$$= \frac{1}{16}x^4 - \frac{3}{2}x^3 + \frac{27}{2}x^2 - 54x + 81$$</td></tr>
<tr><td>2</td><td>$$f'(x) = e^x(5-x)^3 + e^x \times -3(5-x)^2$$
$$= e^x(5-x)^3 - 3e^x(5-x)^2$$
$$= e^x(5-x)^2(2-x)$$
$$g(x) = \sin^{-1} u, \quad u = 5x$$
$$g'(x) = \frac{1}{\sqrt{1-u^2}} \times 5 = \frac{5}{\sqrt{1-25x^2}}$$</td></tr>
<tr><td>3</td><td>$$Inverse = \frac{1}{10+2\alpha}\begin{pmatrix} 2 & -\alpha \\ 2 & 5 \end{pmatrix}, \quad 10 + 2\alpha = 0, \; x = -5$$</td></tr>
<tr><td>4</td><td>By substitution $i^4 + 4i^3 + 3i^2 + 4i + 2 = 0 \quad \rightarrow 1 - 4i - 3 + 4i + 2 = 0$

Hence $z = i$ is a root and $z = -i$ is also a root

Thus $(z+i)(z-i) = z^2 + 1$ is a factor of the original polynomial.

Algebraic long division gives factors of $(z^2+1)(z^2+4z+2) = 0$
$$\rightarrow z^2 + 4z + 2 = 0, \quad (z+2)^2 - 2 = 0, \; z = -2 \pm \sqrt{2}$$
Thus the solutions are $z = \pm i, \; z = 2 \pm \sqrt{2}$</td></tr>
<tr><td>5</td><td>$y = |f(x)|$ $\qquad\qquad$ $y = |f(x) + 2|$



</td></tr>
</table>

6	$\frac{dy}{dt} = -2\sin 2t\ ,\ \frac{dx}{dt} = \ 2\cos 2t,$ $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx},\qquad \frac{dy}{dx} = -\frac{(2\sin 2t)}{2\cos 2t} = \ -\mathbf{\tan 2}\boldsymbol{t}$ $\frac{d^2y}{dx^2} = \frac{d}{dx}(-\tan 2t) \times \frac{dt}{dx},$ $\frac{d^2y}{dx^2} = -2\sec^2 2t \times \frac{1}{2\cos 2t} \quad = -\frac{\sec^2 2t}{\cos 2t} = \mathbf{-\sec^3 2}\boldsymbol{t}$
7	$\int x^2e^{-x}\,dx = \ -x^2e^{-x} - \int -e^{-x} \times 2x\,dx$ $= \ -x^2e^{-x} + \int 2xe^{-x}\,dx$ $= \ -x^2e^{-x} + [-2xe^{-x} - \int -2e^{-x}\,dx]$ $= \ -x^2e^{-x} + [-2xe^{-x} + 2\int e^{-x}\,dx]$ $= \ -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$
8	$U_n = a + d(n-1),$ $U_{20} = 5 + d(20) \rightarrow \quad 62 = 5 + 19d,\ \ d = 3$ $S_n = \frac{1}{2}n\big(2a + d(n-1)\big) \qquad S_{40} = \frac{1}{2} \times 40\big(10 + 3(39)\big) = 2540$
9	Identify the locus in the complex plane given by $\lvert z + i\rvert = 5$ $\lvert z + i\rvert = 5$ $\lvert x + iy + i\rvert = \lvert 5\rvert$ $\lvert x + i(y+1)\rvert = 5$ $\boldsymbol{x^2 + (y+1)^2 = 5^2}$ This a circle with a centre $(0, -1)$and a radius of 5 units
10	$\begin{pmatrix} 1 & -1 & 1: & 1 \\ 2 & 1 & -3: & -6 \\ 1 & -1 & 2: & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & -1 & 1: & 1 \\ 0 & 3 & -5: & -8 \\ 0 & 0 & 1: & 1 \end{pmatrix} \ R_2 - 2R_1,\ \ R_3 - R_1$ $\boldsymbol{c = 1},\quad 3b - 5c = \ -8 \quad 3b = \ -3, \qquad \boldsymbol{b = \ -1}$ $a - b + c = 1,\ \ a + 2 = 1,\ \ \boldsymbol{a = \ -1}$

11	For $u = x^4$, $du = 4x^3dx$ and $\frac{1}{2}du = 2x^3dx$ $$\int \frac{2x^3}{1+x^8}dx = \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\tan^{-1}(x^4) + C$$
12	$2678 = 221 \times 12 + 26$ $221 = 26 \times 8 + 13$ $26 = 13 \times 2 + 0$ $13 = 221 - 26 \times 8$ $13 = 221 - 8(2678 - 221 \times 12)$ $13 = 221 - 8 \times 2678 + 96 \times 221$ $13 = 97 \times 221 - 8 \times 2678$ $a = 97, \quad b = -8$
13	$A^4 = (4A - 3I)(4A - 3I)$ $= 16A^2 - 24AI + 9I$ $= 16A^2 - 24A + 9I$ $= 16(4A - 3I) - 24A + 9I$ $= 64A - 48I - 24A + 9I$ $= 40A - 39I \qquad \boldsymbol{P = 40, \; q = -39}$
14	For $n = 1$, LHS $3(1-1) = 0$ RHS $0(1)(2) = 0$ so true for $n = 1$ Assume that the statement is true for $n = k$ $$\sum_{r=1}^{k} 3(r^2 - r) = (k-1)k(k+1)$$ Now consider $n = k + 1$ $$\sum_{r=1}^{k+1} 3(r^2 - r) = \sum_{r=1}^{k} 3(r^2 - r) + f(k+1)$$ $= (k-1)k(k+1) + 3((k+1)^2 - (k+1))$ $= (k+1)[(k-1)k + 3((k+1) - 1)]$ $(k+1)[(k-1)k + 3k]$ $= (k+1)[(k^2 - k + 3k]$ $= (k+1)[k(k+2)]$ $= k(k+1)(k+2)$ $((k+1) - 1)(k+1)((k+1) + 1)$ So statement is true for $n = k + 1$. Since true for $n = 1, n = k$ and $n = k + 1$, then by induction this is true for all positive integers n $$\sum_{r=11}^{40} 3(r^2 - r) = \sum_{r=1}^{40} 3(r^2 - r) - \sum_{r=1}^{10} 3(r^2 - r)$$ $= (40-1)(40)(40+1) - (10-1)(10)(10+1) = 62970$

15	The auxiliary equation is $m^2 - 2m + 10 = 0, \quad m = \frac{2 \pm \sqrt{-36}}{2} = 1 \pm 3i$ The general solution is $y = e^x\,(A\cos 3x + B\sin 3x)$ $y = 2$ when $x = 0$ $\quad 2 = 1(A + 0), \quad A = 2$ $\frac{dy}{dx} = e^x(A\cos 3x + B\sin 3x) + e^x\,(3B\cos 3x - 3A\sin 3x)$ $\frac{dy}{dx} = 5$ when $x = 0$ $5 = 1(A + 0) + 1\,(0 + 3B), \quad 5 = 2 + 3B, \; B = 1$ $y = e^x\,(2\cos 3x + \sin 3x)$
16	(a) Vertical asymptote at $x = -2$, By polynomial division $g(x) = x - 2 + \frac{9}{x+2}$, there is an oblique asymptote at $y = x - 2$ (b) $\left(0, \frac{5}{2}\right)$ (c) $g'(x) = \frac{2x(x+2)-(x^2+5)}{(x+2)} = \frac{x^2+4x-5}{(x+2)^2} = \frac{(x+5)(x-1)}{(x+2)^2}$, $\frac{(x+5)(x-1)}{(x+2)^2} = 0, \quad x = -5, 1$ Turning points at $(-5,-10), (1,2)$ $g''(x) = \frac{18x+36}{(x+2)^4} = \frac{18}{(x+2)^3}, \quad g''(-5) < 0$ and $g'(1) > 0$ Maximum turning point at $(-5,-10)$ minimum turning point at $(1, 2)$
16d