Lesson 1 – Solving first order differential equations by separating variables

A first order differential equation takes the form $\boldsymbol{f(y)\frac{dy}{dx} = g(x)}$

To find the general solution to this differential equation we use the technique of **separating variables**

Separate the variables

$$f(y)\frac{dy}{dx} = g(x)$$

$$f(y)dy = g(x)dx$$

Integrate both sides

$$\int f(y)dy = \int g(x)dx$$

State explicit or implicit solution

$$F(y) = G(x) + c$$

Where $F(y)$ and $G(x)$ are the antiderivatives of $f(y)$ and $g(x)$

This process will use common forms, integration by substitution and integration by parts. Also watch out for $\int \frac{1}{1+x^2}dx = tan^{-1}x + C$

Example 1 find the general solution to the equation $x\frac{dy}{dx} = 4$

Separate the variables

$$x\frac{dy}{dx} = 4$$

$$dy = \frac{4}{x}dx$$

Integrate both sides

$$\int 1dy = \int \frac{4}{x}dx$$

explicit

$$y = 4\ln x + C$$

Example 2 find the general solution to the equation $\frac{dy}{dx} = 3xy$

Separate the variables
$$\frac{dy}{dx} = 3xy$$

$$\frac{1}{y}\, dy = 3x\, dx$$

Integrate both sides
$$\int \frac{1}{y}\, dy = \int 3x\ dx$$

implicit solution
$$\ln y = \frac{3}{2}x^2 + C$$

explicit solution
$$y = e^{\frac{3}{2}x^2 + C}$$

When given **initial conditions** (values for x and y) you can then find the **particular** solution to a differential equation.

Example 1 find the particular solution to the equation $\frac{dy}{dx} = \frac{y-4}{x+3}$

Given that $y = 20$ when $x = 5$

$$\frac{dy}{dx} = \frac{y-4}{x+3} \;\rightarrow\; \frac{1}{y-4}dy = \frac{1}{x+3}\ dx$$

$$\int \frac{1}{y-4}\ dy = \int \frac{1}{x+3}\ dx$$

$$\ln(y-4) \ = \ln(x+3) + \ C$$

$$\ln 16 = \ln 8 + C, \quad C = \ln 2$$

$$\ln(y-4) = \ln(x+3) + \ \ln 2$$

$$\text{or} \quad \ln(y-4) = \ln(2x+6) \ \rightarrow y = 2x + 10$$

In MIA - Exercise 8.1 Q1 general solutions (not Q1g), Q2a-2i particular solutions

In Leckie and Leckie - Exercise 5A Q1 general solutions, Q2 particular solutions.

Lesson 2 - Modelling real-life situations using differential equations

Example 1 - From Higher Past Paper

The rate of change of the temperature T^0C of a mug of coffee is given by

$$\frac{dT}{dt} = \frac{1}{25}t - k, \qquad 0 \le 5 \le 50$$

- t is the elapsed time in minutes after the coffee is poured into the mug
- k is the constant of proportionality
- Initially the temperature of the coffee is $100\ °C$
- 10 minutes later the temperature has fallen to $82\ °C$

Express T in terms of t.

$$\frac{dT}{dt} = \frac{1}{25}t - k \ \rightarrow \ dT = \frac{1}{25}t - k\ dt$$

$$\int 1\ dT = \int \frac{1}{25}t - k\ dt$$

$$T = \frac{1}{50}t^2 - kt + C$$

When $t = 0$, T is $100\ °C$.

$100 = \frac{1}{50}0^2 - k \times 0 + C,\ \ C = 100$ $\qquad$ $\boldsymbol{T = \frac{1}{50}t^2 - kt + 100}$

When $t = 10$, T is $82\ °C$.

$82 = \frac{1}{50}10^2 - 10k + 100 \ \rightarrow -20 = -10k \ \rightarrow k = 2$

$$\boldsymbol{T = \frac{1}{50}t^2 - 2t + 100}$$

Example 2

The rate of growth for a virus is said to be directly proportional to the number of people infected with the virus. This can be modelled by

$$\frac{dN}{dt} \propto N, \quad \frac{dN}{dt} = kN \quad \text{where } k \text{ is the constant of proportion}$$

Where t is time in days and N is the number of people infected

When $t = 0$ 4 people have been infected.

When $t = 2$ days, 80 people have been infected.

Express n **explicitly** in terms of t and find how long it takes for 50 000 people to be infected.

$$\frac{dN}{dt} = kN \quad \rightarrow \quad \frac{1}{N}dN = k\ dt$$

$$\int \frac{1}{N}\ dN = \int k\ dt$$

$$\ln N = kt + C$$

$$N = e^{kt+c}$$

$$N = e^{kt}e^{c}$$

$$\boldsymbol{N = Ae^{kt}}, \quad \text{where} \quad A = e^{c}$$

When $t = 0$ 4 people have been infected.

$4 = Ae^{0} \ \rightarrow A = 4$ $\qquad\qquad$ $\boldsymbol{N = 4e^{kt}}$

When $t = 2$ days, 80 people have been infected.

$80 = 4e^{2k} \ \rightarrow 20 = e^{2k} \ \rightarrow 20 = (e^{k})^{2} \ \rightarrow 20^{\frac{1}{2}} = e^{k}$ $\qquad$ $\boldsymbol{N = 4 \times 20^{\frac{t}{2}}}$

For 50 000 people to become infected.

$$50000 = 4 \times 20^{\frac{t}{2}} \ \rightarrow \frac{50000}{4} = 20^{\frac{t}{2}} \ \rightarrow \ \log_{20}(12500) = \frac{t}{2} \ \rightarrow \ t = 6.3\ days$$

In MIA **- Exercise 8.2 - Q5 worked through so try some of Q2 – 7**

In Leckie and Leckie - Exercise 5A Q3 to Q8

Lessons 3 & 4

Solving differential equations using the integrating factor method

When solving a first order differential equation in the form

$\frac{dy}{dx} + P(x)y = Q(x)$ where P and Q are functions involving x only.

We can use an integrating factor $I(x) = e^{\int P(x)dx}$ to ease the integration.

For simplicity the notation used will change to $\frac{dy}{dx} + Py = Q,$ and $I = e^{\int P\,dx}$

To solve the differential equation	$\frac{dy}{dx} + \frac{2y}{x} = x$
1. Identify P (the coefficient of y)	$P = \frac{2}{x},$
2. Find the integrating factor $I = e^{\int P\,dx}$	$\int Pdx = 2\int \frac{1}{x}\,dx = 2\ln x$ $I = e^{2\ln x} = e^{\ln x^2} = x^2$
3. Multiply through by the integrating factor	$x^2\frac{dy}{dx} + x^2\frac{2y}{x} = x^2 \times x$ $x^2\frac{dy}{dx} + 2xy = x^3$
4. Integrate both sides Remember that the LHS has been engineered to be $ye^{\int P\,dx} = Iy$	$\int \left(x^2\frac{dy}{dx} + 2xy\right)dx = \int x^3dx$ $Iy = \int x^3dx$ $x^2y = \frac{1}{4}x^4 + c$
5. Divide through by the coefficient of y	$\boldsymbol{y = \frac{1}{4}x^2 + \frac{c}{x^2}}$
The solution is made up of two parts: the **particular solution** which satisfies the original equation and the **complementary function** which contains the constant of integration	
As $\frac{d}{dx}(x^2y) = 2xy + x^2\frac{dy}{dx}$, then $\int \left(x^2\frac{dy}{dx} + 2xy\right)dx = x^2y$	

Remember that when integrating the RHS you might have to use common forms, substitution or integration by parts.

Example 2 Find the general solution to $\frac{dy}{dx} + 2xy = x$

$$P = 2x, \quad \int P dx = \int 2x\, dx = x^2 \qquad I = e^{\int P\, dx} = e^{x^2}$$

$$e^{x^2}\frac{dy}{dx} + 2xye^{x^2} = xe^{x^2}$$

$$\int \left(e^{x^2}\frac{dy}{dx} + 2xye^{x^2}\right) dx = \int xe^{x^2} dx$$

$$Iy = \int xe^{x^2} dx$$

$$e^{x^2}y = e^{x^2} + c = \frac{\frac{1}{2}e^{x^2} + c}{e^{x^2}}$$

$$\boldsymbol{y = \frac{1}{2} + \frac{c}{e^{x^2}}}$$

As $\frac{d}{dx}\left(e^{x^2}y\right) = 2xe^{x^2}y + e^{x^2}\frac{dy}{dx}$, then $\int \left(e^{x^2}\frac{dy}{dx} + 2xye^{x^2}\right) dx = e^{x^2}y$

Example 2 Solve $\frac{dy}{dx} + \frac{y}{2} = e^x$, given the initial conditions $x = 0, \ y = e$

$$P = \frac{1}{2}, \ \rightarrow \int \frac{1}{2}\, dx = \frac{1}{2}x \qquad I = e^{\frac{1}{2}x}$$

$$e^{\frac{1}{2}x}\frac{dy}{dx} + \frac{ye^{\frac{1}{2}x}}{2} = e^x e^{\frac{1}{2}x}$$

$$\int \left(e^{\frac{1}{2}x}\frac{dy}{dx} + \frac{ye^{\frac{1}{2}x}}{2}\right) dx = \int e^{\frac{3}{2}x}\, dx$$

$$Iy = \int e^{\frac{3}{2}x}\, dx$$

$$e^{\frac{1}{2}x}y = \frac{2}{3}e^{\frac{3}{2}x} + C$$

$$e^0 e = \frac{2}{3}e^0 + C \ \rightarrow \ C = e - \frac{2}{3}$$

$$\boldsymbol{y = \frac{2e^{\frac{3}{2}x} + 3e - 2}{3e^{\frac{1}{2}x}} \quad or \quad y = \frac{1}{3}\left(2e^x + 3e^{\frac{1}{2}x} - e^{-\frac{1}{2}x}\right)}$$

In MIA - Exercise 8.3 - Half of Q1,2 Q4 – 7 for extension
In Leckie and Leckie - Exercise 5B Q1 to Q8

Lessons 5 & 6 Solving Second order differential equations

Second order differential equations take the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = Q(x)$

If $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ you have **homogeneous equations**

To find the general or particular solutions to a second order homogeneous DE:

1. Identify the auxiliary equation
2. Solve the auxiliary equation
3. Express your general solution in one of the three forms
4. For a particular solution, two initial conditions must be given

   Condition 1 $y = b$ when $x = a$ Condition 2 $\frac{dy}{dx} = c$ when $x = a$

For a second order DE in the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ Then the auxiliary equation is a quadratic in the form $am^2 + bm + c = 0$		
If $b^2 - 4ac > 0$	If $b^2 - 4ac = 0$	If $b^2 - 4ac < 0$
Then there are two real, distinct roots. Factorise and solve for m_1 and m_2	Then the roots are coincident Use $m = -\frac{b}{2a}$	Then the roots are non-real Use the quadratic formula $m = p \pm qi$
$\boldsymbol{y = Ae^{m_1x} + Be^{m_2x}}$	$\boldsymbol{y = Ae^{mx} + Bxe^{mx}}$	$\boldsymbol{y = e^{px}(A\cos qx + B\sin qx)}$

Example 1

Find the general solution to $2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 3y = 0$
The auxiliary equation is $2m^2 - 5m - 3 = 0, \quad b^2 - 4ac = 1$ $(2m+1)(m-3) = 0$ $m_1 = -\frac{1}{2}, \quad m_2 = 3,$ Thus the general solution is $\boldsymbol{y = Ae^{-\frac{1}{2}x} + Be^{3x}}$

Example 2 Find the particular solution to $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

given the initial conditions $x = 0,\ y = 1$ and $\frac{dy}{dx} = 0$

The auxiliary equation is $m^2 - 4m + 4 = 0, \quad b^2 - 4ac = 0$

Thus $m = \dfrac{4}{2} = 2$ and the general solution is $y = Ae^{2x} + Bxe^{2x}$

$x = 0, y = 1 \qquad 1 = Ae^0 + 0 \ \rightarrow \ 1 = A$

$x = 0, \dfrac{dy}{dx} = 0 \qquad \dfrac{dy}{dx} = 2Ae^{2x} + Be^{2x} + 2Bxe^{2x}$

$0 = 2A + B \qquad \rightarrow \qquad B = -2$

The particular solution is $\boldsymbol{y = e^{2x} - 2xe^{2x}}$

Example 3 Find the particular solution to $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 10y = 0$

given the initial conditions $x = 0,\ y = 1$ and $\frac{dy}{dx} = 0$

The auxiliary equation is $m^2 - 6m + 10 = 0, \quad b^2 - 4ac = -4$

$m = \dfrac{6 \pm \sqrt{4}}{2}, \quad m = 3 \pm i \qquad y = e^{3x}\,(A\cos x + B\sin x)$

$x = 0, y = 1 \qquad 1 = A\cos 0 + B\sin 0 \qquad 1 = A$

$$\frac{dy}{dx} = 3Ae^{3x}\cos x - Ae^{3x}\sin x + 3Be^{3x}\sin x + Be^{3x}\cos x$$

$x = 0, \frac{dy}{dx} = 0 \qquad 0 = 3A - 0 + 0 + B \ B = -3$

The particular solution is $\boldsymbol{y = e^{3x}(\cos x - 3\sin x)}$

In MIA textbook ***Exercise 8.4 where roots are real and distinct***

Exercise 8.5 where roots are real and coincident

Exercise 8.6 where roots are non-real

Exercise 8.7 mixture of roots

In Leckie and Leckie – Exercise 5C, do the whole exercise.

Lesson 7 Heterogeneous second-order differential equations $(\neq \mathbf{0})$

When given a second order differential equation in the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = Q(x)$$

Then general solution to this non-homogeneous equation is $y = g(x) + k(x)$

Where - $g(x)$ is referred to as the **particular integral PI**

- $k(x)$ is called the **complementary function CF**

The **complementary function CF** comes from the solution to the auxiliary equations (see homogeneous second order DE)

The **particular integral PI** is found by:

1. guessing that $y = g(x)$ will take the same general shape as $Q(x)$
2. Calculating the derivative and second derivative of $g(x)$
3. Equating coefficients to establish an equation for $g(x)$

For $\quad a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c = Q(x)$	
For the complementary function CF	**For the particular integral PI**
For the auxiliary equation $am^2 + bm + c = 0$ If roots are real and distinct $y = Ae^{m_1x} + Be^{m_2x}$ If roots are real and coincident $y = Ae^{mx} + Bxe^{mx}$ If roots are non-real $y = e^{px}\,(A\cos qx + B\sin qx)$	If $Q(x)$ is a linear function Try $y = ax + b$ If $Q(x)$ is a quadratic function Try $y = ax^2 + bx + c$ If $Q(x)$ is an exponential function Try $y = ae^{bx}$ If $Q(x)$ is a trig function Try $y = a\cos x + b\sin x$

Example 1 Find the general solution to $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = 20x - 4$

For the complementary function CF

The auxiliary equation is $m^2 - 7m + 10 = 0$

$$(m-5)(m-2) = 0, \; m = 5 \text{ and } m = 2$$

CF is $\boldsymbol{y = Ae^{5x} + Be^{2x}}$

For the particular integral PI

$20x - 4$ is a linear function, $y = cx + d$, $\frac{dy}{dx} = c$, $\frac{d^2y}{dx^2} = 0$

Substitution into the original equation

$$0 - 7a + 10(cx + d) = 20x - 4$$

$$-7c + 10d = -4, \; 10c = 20 \; \rightarrow c = 2, \quad d = 1$$

PI is $\boldsymbol{y = 2x + 1}$

The general solution is $\boldsymbol{y = Ae^{5x} + Be^{2x} + 2x + 1}$

Example 2 Find the general solution to $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 30\sin x$

For the complementary function CF

The auxiliary equation is $m^2 - 2m + 5 = 0, \quad m = 1 \pm 2i$

CF is $\boldsymbol{y = e^x (A\cos 2x + B\sin 2x)}$

For the particular integral PI

$$y = d\cos x + e\sin x, \quad \frac{dy}{dx} = e\cos x - d\sin x,$$

$$\frac{d^2y}{dx^2} = -d\cos x - e\sin x$$

Substitution into the original equation gives

$$-d\cos x - e\sin x - 2(e\cos x - d\sin x) + 5(d\cos x + e\sin x) = 30\sin x$$

$$(-d - 2e + 5d)\cos x + (-e + 2d + 5e)\sin x = 30\sin x$$

$$4d - 2e = 0, \quad 2d + 4e = 30, \quad e = 6, \quad d = 3$$

PI is $\boldsymbol{y = 3\cos x + 6\sin x,}$

The general solution is $\boldsymbol{y = e^x (A\cos 2x + B\sin 2x) + 3\cos x + 6\sin x,}$

In MIA - Exercise 8.8 - Try the whole exercise (in each case a suggested PI is given)

In Leckie and Leckie – Exercise 5D Q1 is find the general solution

Lesson 8 Identify the particular solution to a non-homogeneous 2ODE 0)

1. Identify the complimentary function $\left(k(x)\right)$using the auxiliary equation
2. Identify the particular integral $\left(g(x)\right)$
3. State the general solution in the form $y = k(x) + g(x)$
4. Use the particular conditions given (including the derivative of the general solution) to identify any coefficients in the general solution

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Example 1 Find the particular solution to $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x^2 + 1$

given the initial conditions $x = 0$, $y = 5$ and $\frac{dy}{dx} = 3$

For the complementary function CF

the auxiliary equation is $m^2 - 3m + 2 = 0, \quad (m-2)(m-1) = 0$

CF is $\boldsymbol{y = Ae^{2x} + Be^{x}}$

For the particular integral PI - using $\boldsymbol{y = ax^2 + bx + c}$

$$y = ax^2 + bx + c, \qquad \frac{dy}{dx} = 2ax + b, \qquad \frac{d^2y}{dx^2} = 2a$$

Substitution into the original equation gives

$$2a - 6ax - 3b + 2ax^2 + 2bx + 2c = 2x^2 + 1$$

$$2ax^2 = 2x^2, \qquad 2bx - 6ax = 0, \qquad 2a + 2c - 3b = 1$$

$$a = 1,\ 2b - 6 = 0, \qquad b = 3, \qquad 2 - 9 + 2c = 1, \qquad c = 4$$

PI is $\boldsymbol{y = x^2 + 3x + 4,}$

The general solution is

$$\boldsymbol{y = Ae^{2x} + Be^{x} + x^2 + 3x + 4,}$$

For the particular solution where $x = 0, \quad y = 5$ and $\frac{dy}{dx} = 3$

$$5 = A + B + 4, \quad 1 = A + B$$

$$\frac{dy}{dx} = 2Ae^{2x} + Be^{x} + 2x + 3$$

$$3 = 2A + B + 3, \quad 0 = 2A + B, \quad A = -1,\ B = 2$$

The particular solution is $\boldsymbol{y = 2e^{x} - e^{2x} + x^2 + 3x + 4,}$

In MIA - Exercise 8.9 - Q1 and 2 is sufficient, Q3 not necessary

In Leckie and Leckie – Exercise 5D Q2 is find the particular solution

Be careful with your initial choice for the PI. Ideally the PI takes the same form as $Q(x)$ – so if $Q(x) = e^x$, then try $y = ae^x$. However, if after substitution we end up with $0 = Q(x)$ then try $y = xQ(x)$ or $y = x^2Q(x)$.

For example

If CF = $Ae^{3x} + Be^{2x}$ and $Q(x) = 2e^{2x}$ try $y = axe^{2x}$ as $y = e^{2x}$ is in the CF

If CF = $Ae^{2x} + Bxe^{2x}$ and $Q(x) = 3e^{2x}$, try $y = ax^2e^{2x}$ as both $ae^{2x}$and axe^{2x} are both in the CF

Try not to arrive at $0 \neq Q(x)$ and if you do, then rethink your choice for PI

2013 Q14 Solve the differential equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 4e^{3x}$

Given that $y = 1$ and $\frac{dy}{dx} = -1$, when $x = 0$

For the complementary function CF

the auxiliary equation is $m^2 - 6x + 9 = 0$, $(m - 3)^2 = 0$

CF is $\boldsymbol{y = Ae^{3x} + Bxe^{3x}}$

For the particular integral PI

$y = Ae^{3x}$ and $y = Bxe^{3x}$ are in CF so try $y = Cx^2e^{3x}$

$$y = Cx^2e^{3x}, \qquad \frac{dy}{dx} = 2Cxe^{3x} + 3Cx^2e^{3x},$$

$$\frac{d^2y}{dx^2} = 2Ce^{3x} + 6Cxe^{3x} + 6Cxe^{3x} + 9Cx^2e^{3x}$$

Substitution into the original equation gives

$$2Ce^{3x} + 6Cxe^{3x} + 6Cxe^{3x} + 9Cx^2e^{3x} - 12Cxe^{3x} - 18Cx^2e^{3x} + 9Cx^2e^{3x} = 4e^{3x}$$

$2Ce^{3x} = 4e^{3x}$, $C = 2$ **Thus PI is** $\boldsymbol{y = 2x^2e^{3x}}$

The general solution is $\boldsymbol{y = Ae^{3x} + Bxe^{3x} + 2x^2e^{3x}}$

For the particular solution where $x = 0$, $y = 1$ and $\frac{dy}{dx} = -1$

$$1 = Ae^0 + 0 + 0, \qquad A = 1$$

$$\frac{dy}{dx} = 3Ae^{3x} + Be^{3x} + 3Bxe^{3x} + 4xe^{3x} + 6x^2e^{3x}$$

$$-1 = 3A + B, \qquad -4 = B,$$

The particular solution is $\boldsymbol{y = e^{3x} - 4xe^{3x} + 2x^2e^{3x}}$