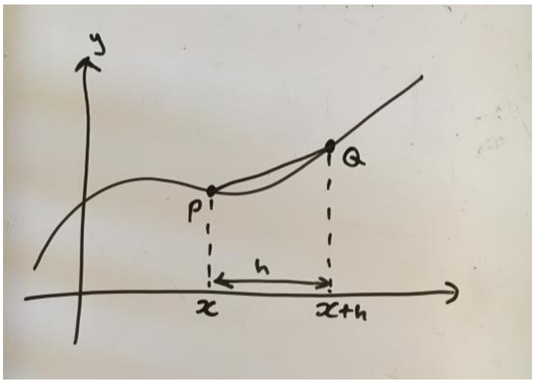
Differentiation from First Principles

- State the definition of a derivative
- State the limit of a simple function as a variable tends to zero
- Prove the derivative of simple functions

Definition of a Derivative



The graph shows the function $y = f(x)$. We can find the gradient of the straight line joining P and Q using the gradient formula:

$$m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P}$$

$$m_{PQ} = \frac{f(x+h) - f(x)}{x + h - x}$$

$$m_{PQ} = \frac{f(x+h) - f(x)}{h}$$

As Q moves closer and closer to P along the curve, the gradient of the straight line PQ approaches the gradient of the curve itself at point P *i.e*, its derivative. The derivative is therefore given by:

$$\lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

Example

Prove that the derivative of $f(x) = x^2$ is $2x$.

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

$$= \frac{(x+h)^2 - x^2}{h}$$

$$= \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \frac{2xh + h^2}{h}$$

$$= \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= 2x + h$$

$$\lim_{h \to 0} (2x+h) = 2x \quad \text{Thus } f'(x) = 2x$$

Exercise

1. $f(x) = 3x$ 2. $f(x) = 4x + 5$ 3. $f(x) = 5x^2$ 4. $f(x) = 2x^2 - 1$

5. $f(x) = x^3$ 6. $f(x) = 4x^3 + 5x$ 7. $f(x) = \frac{1}{x}, x \neq 0$ 8. $f(x) = \frac{3}{x^2}, x \neq 0$