Prelim Examination 2005 / 2006
(Assessing Units 1 & 2)

MATHEMATICS

Higher Grade - Paper I (Non~calculator)

Time allowed - 1 hour 10 minutes

Read Carefully

1. **Calculators may not be used in this paper.**
2. Full credit will be given only where the solution contains appropriate working.
3. Answers obtained by readings from scale drawings will not receive any credit.
4. **This examination paper contains questions graded at all levels.**

FORMULAE LIST

Circle:

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

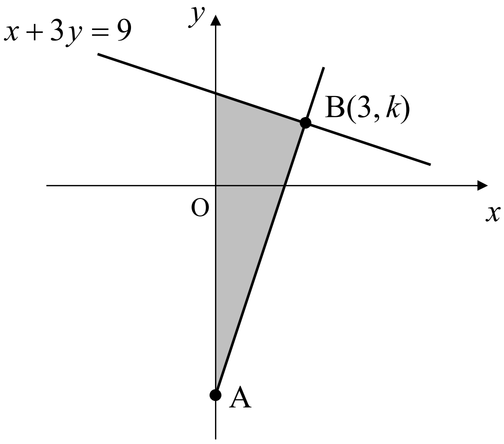
The equation $(x-a)^2 + (y-b)^2 = r^2$ represents a circle centre (a, b) and radius r.

Trigonometric formulae:

$$
\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\sin 2A &= 2 \sin A \cos A \\
\cos 2A &= \cos^2 A - \sin^2 A \\
&= 2\cos^2 A - 1 \\
&= 1 - 2\sin^2 A
\end{aligned}
$$

All questions should be attempted

1. Part of the line with equation $x + 3y = 9$ is shown in the diagram. B lies on this line and has coordinates $(3, k)$.



(a) Find the value of k. **1**

(b) Given that the line AB is perpendicular to the line $x + 3y = 9$, find the equation of the line AB. **3**

(c) Hence write down the coordinates of A. **1**

(d) Calculate the area of the shaded triangle. **4**

2. (a) A function f has as its derivative $f'(x) = x^3 - ax^2 - 4ax$.

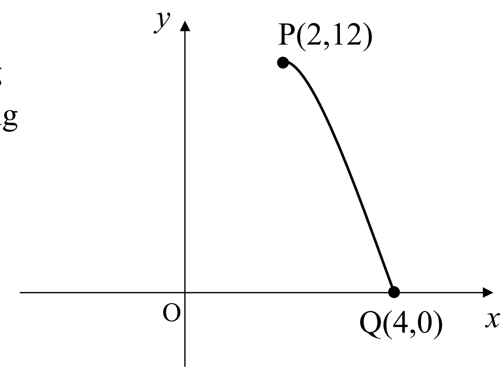
Find a if the function has a stationary point at $x = 4$. **4**

(b) Hence find the rate of change of this function at $x = -2$ and comment on your result. **2**

3. A quadratic function, defined on a suitable domain, is given as $f(x) = 12x - 3x^2$.

The diagram shows part of the graph of this quadratic function, $y = f(x)$.

The graph passes through the points P(2,12) and Q(4,0) as shown.



(a) Sketch the graph of $y = -f(x) + 6$ marking clearly the image points of P and Q and stating their coordinates. **3**

(b) Given that $g(x) = -f(x) + 6$, write down a formula for $g(x)$. **2**

4. Find a given that $\int_a^2 (4 + 2x)\ dx = 0$, where $a < 2$. **5**

5. Find all the values of x in the interval $0 \le x \le 2\pi$ for which $\sin 2x = -\cos x$. **6**

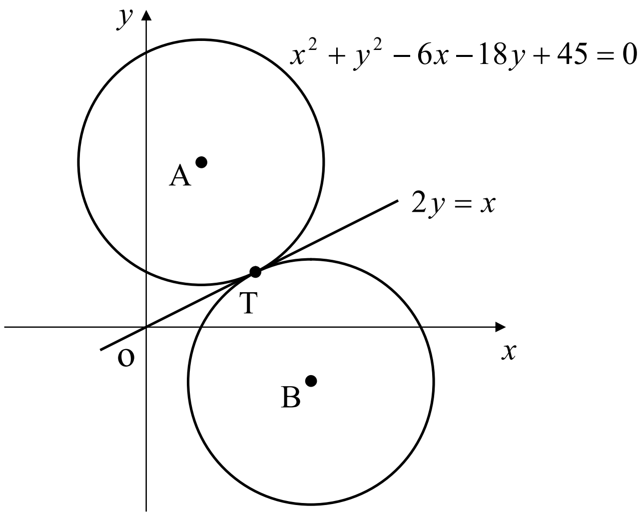
6. Two functions are defined on suitable domains as $f(x) = x + 1$ and $g(x) = x^2 + 6x + 13$.

Given that the function h is such that $h(x) = g(f(x))$, express h in the form

$$h(x) = (x + a)^2 + b, \text{ where } a \text{ and } b \text{ are integers,}$$

and hence write down the minimum value of h and the corresponding replacement for x. **6**

7. The diagram below shows two congruent circles which touch at a single point T.
The circle, centre A, has as its equation $x^2 + y^2 - 6x - 18y + 45 = 0$.
The line with equation $2y = x$ is the common tangent to the two circles through T.



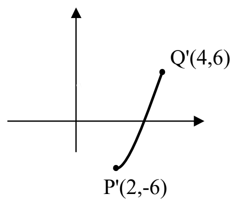
(a) **Show algebraically** that T has coordinates (6,3). **5**

(b) Hence establish the the coordinates of B, the centre of the lower circle, and find the equation of this circle. **6**

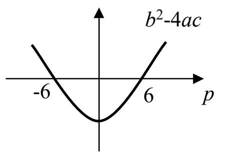
8. Find $f'(x)$ when $f(x) = \dfrac{x^2 - 2\sqrt{x}}{x}$, expressing your answer with positive indices, and hence calculate the value of the gradient of the tangent to the curve $y = f(x)$ at $x = \frac{1}{4}$. **6**

9. What can you say about p if the equation, in x, $\dfrac{x}{p} + \dfrac{9}{px} = 1$ has **no real** roots ? **6**

[END OF QUESTION PAPER]

	Give 1 mark for each •	Illustration(s) for awarding each mark
1.	**(a) ans:** k = 2 **1 mark** •1 sub. to answer **(b) ans:** $y = 3x - 7$ (or equiv.) **3 marks** •1 for gradient of line •2 for gradient of AB •3 for equation **(c) ans:** A(0,-7) **1 mark** •1 answer (y intercept) **(d) ans:** Area = 15 square units **4 marks** •1 y intercept of top line •2 length of base •3 perpendicular length •4 calculation to answer	(a) •1 $3 + 3k = 9 \;\therefore\; k = 2$ (b) •1 $m = -\frac{1}{3}$ •2 $m_{AB} = 3$ •3 $y - 2 = 3(x - 3)$ (c) •1 A(0,-7) (d) •1 (0,3) •2 distance between y intercepts = 10 •3 y -axis to B … 3 units •4 $A = \frac{1}{2}bh$ $= \frac{1}{2} \times 10 \times 3 = 15 \; units^2$
2.	**(a) ans:** a = 2 **4 marks** •1 knowing to solve deriv. to zero •2 setting up synthetic division •3 completing synth. div. •4 solving equ. to zero and answer **(b) ans:** zero, another stat. point **2 marks** •1 substituting for a and -2 •2 calculation finds zero + conclusion ** *pupils may complete part (a) by substitution.*	(a) •1 at s.p. $f'(x) = 0$ (stated or implied) •2 4 | 1 -a -4a 0 •3 4 | 1 -a -4a 0 4 16-4a 64-32a 1 4-a 16-8a 0 •4 64-32a = 0 , a = 2 (b) •1 …… ….. $f'(-2) = (-2^3) - 2(2)(-2^2) - 4(2)(-2)$ •2 $f'(-2) = 0 \;\therefore$ another stat. point
3.	**(a) ans:** see sketch **3 marks** •1 for reflecting •2 for translation up 6 •3 for annotating, coordinates **(b) ans:** $g(x) = 3x^2 - 12x + 6$ **2 marks** •1 for strategy •2 for answer (any equivalent form)	(a) •1 •2 •3  (b) •1 strategy •2 $g(x) = -(12x - 3x^2) + 6$

	Give 1 mark for each •	Illustration(s) for awarding each mark
4.	**ans:** $a = -6$ **5 marks** •1 integrating first term •2 integrating 2^{nd} term •3 sub. in limits •4 simplifying to quad. equation •5 solving and choosing answer	•1 $4x$ …… •2 ….. $+\frac{2x^2}{2}$ •3 $(8+4)-(4a+a^2)=0$ •4 $a^2+4a-12=0$ •5 $(a-2)(a+6)=0$ ~~$a=2$~~ or $a=-6$
5.	**ans:** $\{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}\}$ **6 marks** •1 knowing to solve to zero •2 replacement •3 factorising and 2 solutions •4 1^{st} angle from one solution •5 1^{st} angle from other solution •6 for remaining two angles	•1 $\sin 2x + \cos x = 0$ •2 $2\sin x \cos x + \cos x = 0$ •3 $\cos x(2\sin x + 1) = 0$ $\therefore$ $\cos x = 0$ or $\sin x = -\frac{1}{2}$ •4 $x = \frac{\pi}{2}$ •5 $x = \frac{7\pi}{6}$ •6 $x = \frac{3\pi}{2}, \frac{11\pi}{6}$
6.	**ans:** $h(x) = (x+4)^2 + 4$, $h_{min} = 4$ @ $x = -4$ **6 marks** •1 for *f* into g •2 for expansion and simplifying •3 bracket term •4 number term •5 for minimum value •6 for *x*	•1 $g(f(x)) = (x+1)^2 + 6(x+1) + 13$ •2 $h(x) = x^2 + 8x + 20$ •3 [$(x+4)^2$...... •4 ….. -16] $+20$ •5 min = 4 •6 @ x = -4

	Give 1 mark for each •	Illustration(s) for awarding each mark
7.	**(a) ans:** proof **5 marks** •1 know to solve a system •2 combining equations •3 simplifying to quad. •4 for 1st coordinate •5 for 2nd coordinate **(b) ans:** B(9,-3) , $(x-9)^2+(y+3)^2=45$ **6 marks** •1 knowing T mid-pt between centres •2 drawing out centre of top circle •3 finding B •4 knowing r the same •5 finding r^2 •6 writing down equation of lower circle	(a) •1 set up a system •2 $(2y)^2+y^2-6(2y)-18y+45=0$ •3 $5y^2-30y+45=0$ •4 $5(y-3)(y-3)=0 \ \therefore\ y=3$ •5 $x=2(3)=6$ (or equivalent) (b) •1 strategy •2 A(3,9) •3 A(3,9) $\rightarrow$T(6,3) $\rightarrow$B(9,-3) •4 stated or implied $r_1=r_2$ •5 $r^2=\sqrt{9+81-45}=45$ •6 $(x-9)^2+(y+3)^2=45$
8.	**ans:** $f'(x)=1+\frac{1}{x^{\frac{3}{2}}}$, 9 **6 marks** •1 preparing to differentiate •2 diff. 1st term •3 diff. 2nd term •4 writing with positive indices •5 substituting •6 answer	•1 $f(x)=x^{-1}(x^2-2x^{\frac{1}{2}})$ $=x-2x^{-\frac{1}{2}}$ •2 1...... •3 .... $+1x^{-\frac{3}{2}}$ •4 $f'(x)=1+\frac{1}{x^{\frac{3}{2}}}$ •5 $f'(\frac{1}{4})=1+\frac{1}{(\frac{1}{4})^{\frac{3}{2}}}$ •6 $f'(\frac{1}{4})=1+\frac{1}{\frac{1}{8}}=9$
9.	**ans:** $-6<p<6$ (or equivalent) **6 marks** •1 dealing with the fractions •2 manipulation to quad. form •3 discriminant statement •4 for a, b and c •5 finding discriminant •6 solution from quad. inequat.	•1 strategy .... $\times px$ (or equiv.) •2 $x^2+9=px$ $x^2-px+9=0$ •3 for no real roots $b^2-4ac<0$ •4 $a=1$, $b=-p$, $c=9$ •5 $p^2-36<0$ •6 $-6<p<6$ 

Total 60 marks

Prelim Examination 2005 / 2006
(Assessing Units 1 & 2)

MATHEMATICS

Higher Grade - Paper II

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Trigonometric formulae:

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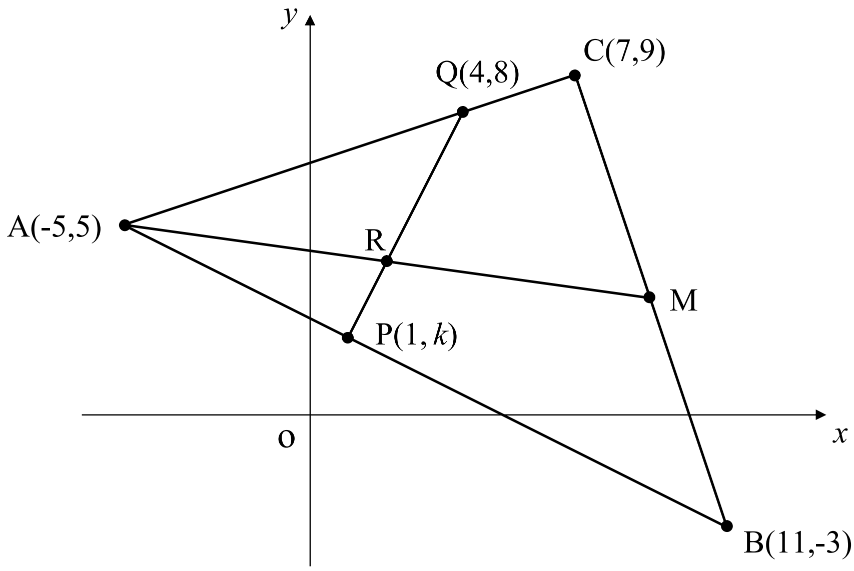
&= 1 - 2\sin^2 A
\end{aligned}$$

All questions should be attempted

1. Triangle ABC has vertices A(-5,5), B(11,-3) and C(7,9).
Q(4,8) lies on AC and AM is a median of the triangle.



(a) Given that A, P and B are collinear, find the value of k. **4**

(b) Hence find the equation of PQ. **2**

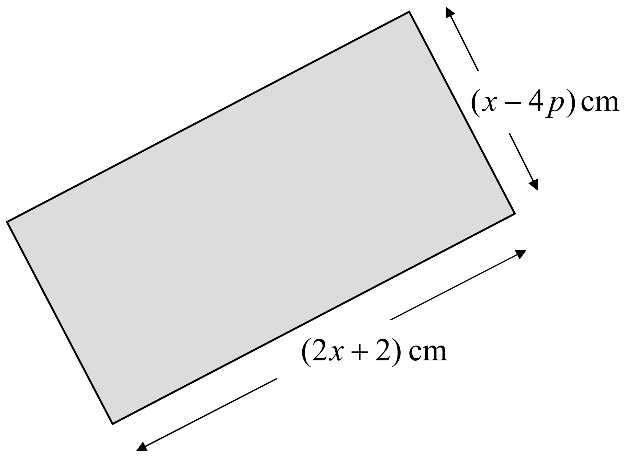
(c) Find the coordinates of R, the point of intersection between the line PQ and the median AM. **5**

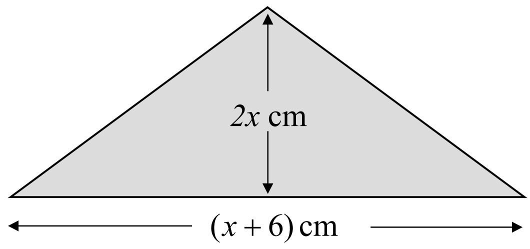
2. A scientist studying a large colony of bats in a cave has noticed that the fall in the population over a number of years has followed the recurrence relation $U_{n+1} = 0 \cdot 75U_n + 200$, where n is the time in years and 200 is the average number of bats born each year during a concentrated breeding week.

(a) He estimates the colony size at present to be 2100 bats with the breeding week just over. Calculate the estimated bat population in 4 years time immediately **before** that years breeding week. **3**

(b) The scientist knows that if in the **long term** the colony drops, at any time, below 700 individuals it is in serious trouble and will probably be unable to sustain itself.
Is this colony in danger of extinction?
Explain your answer with words and appropriate working. **4**

3. The diagram below shows a rectangle and an isosceles triangle.
The letter p is a constant. All lengths are in centimetres.





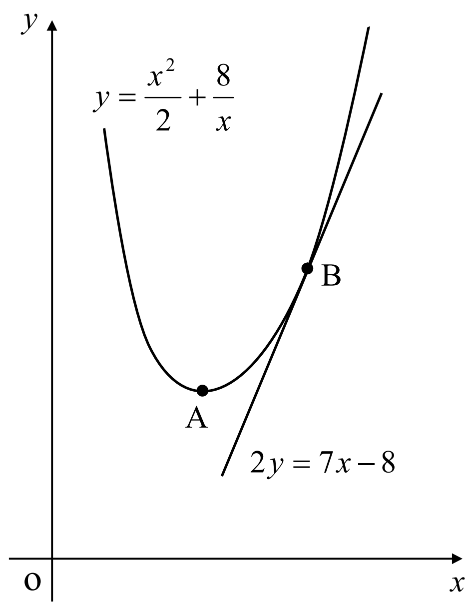
(a) Taking A_1 as the area of the rectangle, and A_2 as the area of the triangle, show clearly that the difference between the two areas can be written in the form

$$A_1 - A_2 = x^2 - (8p+4)x - 8p$$ **4**

(b) Given that $A_1 - A_2 = 1\,\text{cm}^2$, establish the value of p, where p is > -1, for this equation to have **only one solution** for x. **6**

(c) Hence find x when p takes this value. **2**

4. The diagram shows part of the graph of the curve with equation $y = \dfrac{x^2}{2} + \dfrac{8}{x}$, $x \neq 0$.



(a) Find the coordinates of the stationary point A. **5**

(b) Also shown is the line with equation $2y = 7x - 8$ which is a tangent to the curve at B.
Establish the coordinates of B. **6**

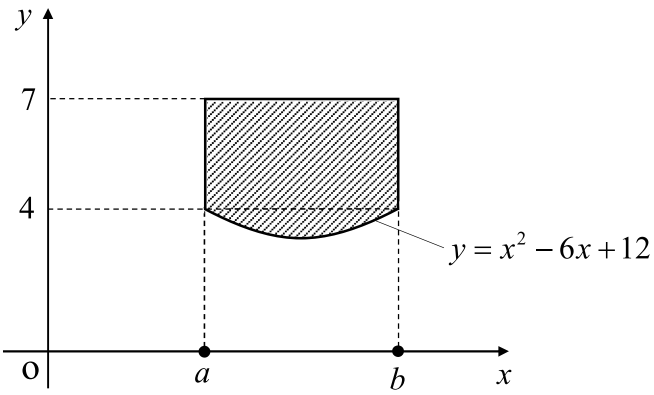
5. A function is defined on a suitable domain as $f(x) = x^2 - a$, where a is a constant.

(a) Find a formula for h given that $h(x) = f(f(x))$. **2**

(b) Given now that $h'(2) = 8$, find a. **3**

6. The diagram, which is not drawn to scale, shows the cross-section of an iron bar. The units are in centimetres.

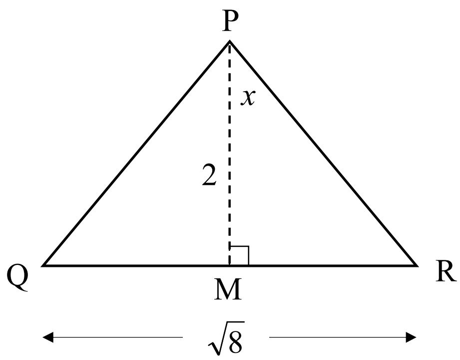
When placed in the coordinate diagram the curved section of the rod has as its equation $y = x^2 - 6x + 12$.



(a) **Show algebraically** that the values of a and b are 2 and 4 respectively. **2**

(b) Calculate the shaded area in square centimetres. **6**

7. Triangle PQR is isosceles with PQ = PR as shown. M is the midpoint of QR. QR = $\sqrt{8}$ units, PM = 2 units and $\angle$RPM = x.

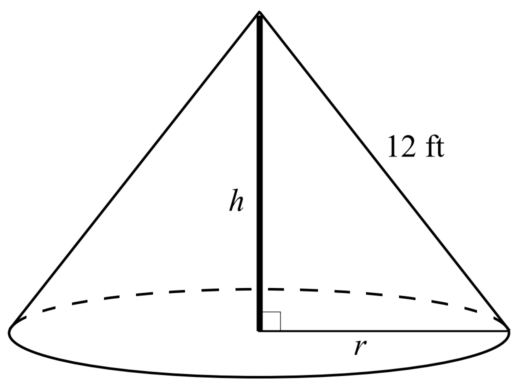


(a) Show clearly that the exact value of $\sin x$ is $\frac{1}{3}\sqrt{3}$. **4**

(b) Hence, or otherwise, show that $\sin \angle RPQ = \frac{2}{3}\sqrt{2}$. **4**

8. An old fashioned bell tent is in the shape of a cone.

The tent has radius r, vertical height h and a slant height of 12 feet as shown.



(a) Write down an expression for r^2 in terms of h. **1**

(b) Hence show clearly that a function, in terms of h, for the volume of this cone can be expressed as

$$V(h) = 48\pi h - \tfrac{1}{3}\pi h^3$$

[note: the formula for the volume of a cone is $V = \tfrac{1}{3}\pi r^2 h$] **2**

(c) Hence find the exact value of h, the height of the tent, which would maximise the volume of the tent.
Justify your answer. **5**

[END OF QUESTION PAPER]

	Give 1 mark for each •	Illustration(s) for awarding each mark
1.	**(a) ans:** $k = 2$ **4 marks** •1 gradient of AB •2 gradient of AP •3 equating gradients •4 finding k **(b) ans:** $y = 2x$ **2 marks** •1 gradient of PQ •2 equation **(c) ans:** R(2,4) **5 marks** •1 coordinates of M •2 equation of median •3 setting up a system •4 finding first coordinate •5 finding 2nd coordinate	(a) •1 $m_{AB} = \frac{-3-5}{11+5} = -\frac{1}{2}$ •2 $m_{AP} = \frac{k-5}{6}$ •3 $\frac{k-5}{6} = -\frac{1}{2}$ •4 $k = 2$ (b) •1 $m_{PQ} = \frac{8-2}{4-1} = 2$ •2 $y - 2 = 2(x-1)$ (c) •1 M(9,3) *no mark given for gradient of AM* … $m_{AM} = -\frac{1}{7}$ •2 $y - 3 = -\frac{1}{7}(x-9)$ •3 $7y = -x + 30$ $y = 2x$ •4 $x = 2$ •5 $y = 4$
2.	**(a) ans:** $1011 \cdot 33$ bats (ignore rounding) **3 marks** •1 first two lines of calculation •2 lines 3 and 4 of calculations •3 answer **(b) ans:** Colony is in danger. 600 prior to breeding week is less than 700 bats **4 marks** •1 knows to calculate limit + knows formula •2 calculates limit correctly •3 knows to subtract 200 •4 explanation	(a) •1 …… Low High $U_1 = 0 \cdot 75(2100) = 1575 + 200 = 1775$ $U_2 = 0 \cdot 75(1775) = 1331 \cdot 25 + 200 = 1531 \cdot 25$ •2 …… $U_3 = 0 \cdot 75(1531 \cdot 25) = 1148 \cdot 24 + 200 = 1348 \cdot 44$ $U_4 = 0 \cdot 75(1348 \cdot 44) = 1011 \cdot 33$ •3 $1011 \cdot 33$ (b) •1 $L = \frac{b}{1-a}$ •2 $L = \frac{200}{1 - 0 \cdot 75} = 800$ •3 low population 800 - 200 =600 •4 600 prior to breeding week is less than 700 bats so colony in danger

	Give 1 mark for each •	Illustration(s) for awarding each mark
3.	**(a) ans:** proof **4 marks** •1 area of rectangle •2 area of triangle •3 subtracting areas •4 tidy up and common factor **(b) ans:** $p=-\frac{1}{4}$ **6 marks** •1 equating to zero •2 discriminant statement •3 a, b and c •4 substitution •5 to quadratic form •6 answer **(c) ans:** $x=1$ **2 marks** •1 substitution •2 solving to answer	(a) •1 $A_{rec}=(2x+2)(x-4p)$ $=2x^2+2x-8px-8p$ •2 $A_{tri}=\frac{1}{2}(x+6)\times 2x=x^2+6x$ •3 $A_1-A_2=$ $=2x^2+2x-8px-8p\ -(x^2+6x)$ •4 $A_1-A_2=x^2-(8p+4)x-8p$ (b) •1 $x^2-(8p+4)x-8p-1=0$ •2 $b^2-4ac=0$ for equal roots •3 $a=1,\ b=-(8p+4)$, $c=-8p-1$ •4 $(8p+4)^2-4(-8p-1)=0$ •5 $64p^2+96p+20=0$ •6 $4(4p+5)(4p+1)=0$ ~~$p=-\frac{5}{4}$~~ *or* $p=-\frac{1}{4}$ (c) •1 $x^2-(8(-\frac{1}{4})+4)x-8(-\frac{1}{4})-1=0$ •2 $(x-1)(x-1)=0$, $x=1$
4.	**(a) ans:** A(2,6) **5 marks** •1 knowing and preparing to differentiate •2 differentiating •3 solving to zero •4 x coordinate •5 y coordinate **(b) ans:** B(4,10) **6 marks** •1 know to form a system •2 combining equations •3 manipulation to polynomial form •4 sets up synthetic division •5 finds x coordinate •6 for y coordinate	(a) •1 $y=\frac{1}{2}x^2+8x^{-1}$ •2 $\frac{dy}{dx}=x-8x^{-2}=x-\frac{8}{x^2}$ •3 $x-\frac{8}{x^2}=0$ •4 $x^3-8=0\quad\therefore\quad x=2$ •5 $y=6$ (b) •1 $y=\frac{x^2}{2}+\frac{8}{x}$ $2y=7x-8$ •2 $7x-8=x^2+\frac{16}{x}$ •3 $x^3-7x^2+8x+16=0$ •4 $\begin{array}{\|cccc} 1 & -7 & 8 & 16 \\ \hline & & & 0 \end{array}$ •5 $\begin{array}{c\|cccc} 4 & 1 & -7 & 8 & 16 \\ & & 4 & -12 & -16 \\ \hline & 1 & -3 & -4 & 0 \end{array}\quad x=4$ •6 $2y=7(4)-8\quad\therefore\quad y=10$

	Give 1 mark for each •	Illustration(s) for awarding each mark
5.	**(a) ans:** $h(x) = x^4 - 2ax^2 + a^2 - a$ **2 marks** •1 for dealing with composition •2 for formula (any equivalent form) **(b) ans:** $a = 3$ **3 marks** •1 for differentiating •2 substituting and solving to 8 •3 answer	(a) •1 $f(x^2 - a) = ...$ •2 $h(x) = (x^2 - a)^2 - a$ $= x^4 - 2ax^2 + a^2 - a$ (b) •1 $h'(x) = 4x^3 - 4ax$ •2 $4(2^3) - 4(2)a = 8$ •3 $a = 3$
6.	**(a) ans:** proof **2 marks** •1 for solving to 4 •2 for answer **(b) ans:** $7\frac{1}{3}$ cm^2 **6 marks** •1 setting up integral •2 integrating •3 substituting limits •4 area under curve •5 area of rectangle •6 subtraction to answer	(a) •1 $x^2 - 6x + 12 = 4$ •2 $(x-4)(x-2) = 0 \;\therefore\; x = 2 \text{ or } 4$ (b) •1 $A = \int_2^4 x^2 - 6x + 12 \; dx$ •2 $= \left[\frac{x^3}{3} - 3x^2 + 12x\right]_2^4$ •3 $= (\frac{64}{3} - 3(16) + 12(4)) - (\frac{8}{3} - 12 + 24)$ •4 $= 6\frac{2}{3}$ cm^2 •5 $A_{rec} = 2 \times 7 = 14$ •6 $A = 14 - 6\frac{2}{3} = 7\frac{1}{3}$ cm^2

	Give 1 mark for each •	Illustration(s) for awarding each mark
7.	**(a) ans:** proof **4 marks** •1 length of MR •2 length of PR •3 value of $\sin x$ •4 required form **(b) ans:** proof **4 marks** •1 knowing angle is equiv. to $\sin 2x$ •2 use replacement •3 for value of $\cos x$ *and* substitution •4 required answer	(a) •1 MR $= \sqrt{2}$ *or* $\frac{\sqrt{8}}{2}$ •2 $PR^2 = 2^2 + \sqrt{2}^2 = 6 \therefore PR = \sqrt{6}$ •3 $\sin x = \frac{\sqrt{2}}{\sqrt{6}}$ •4 $\sin x = \frac{\sqrt{2}}{\sqrt{6}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$ (b) •1 $\sin RPQ = \sin 2x$ •2 $= 2\sin x \cos x$ •3 $= 2 \times \frac{\sqrt{3}}{3} \times \frac{2}{\sqrt{6}}$ •4 $\frac{4\sqrt{3}}{3\sqrt{6}} = \frac{4}{3\sqrt{2}} = \frac{2}{3}\sqrt{2}$
8.	**(a) ans:** $r^2 = 144 - h^2$ **1 mark** •1 answer **(b) ans:** proof **2 marks** •1 substituting for r^2 •2 required form **(c) ans:** $h = \sqrt{48} = 4\sqrt{3}$ ft **5 marks** •1 know to differentiate •2 solve derivative to zero •3 differentiate •4 solve for h •5 justification	(a) •1 $r^2 = 144 - h^2$ (b) •1 $V = \frac{1}{3}\pi(144 - h^2)h$ •2 $V(h) = 48\pi h - \frac{1}{3}\pi h^3$ (c) •1 strategy to diff. (stated or implied) •2 solve $V'(h) = 0$ (stated or implied) •3 $V'(h) = 48\pi - \pi h^2$ •4 $48\pi - \pi h^2 = 0$ $h^2 = 48$ $h = \sqrt{48} = 4\sqrt{3}$ •5 $\rightarrow$ | $4\sqrt{3}$ | $\rightarrow$ + | 0 | - / | — | \\ max
		Total 70 marks