Advanced Higher Applied Mathematics 2004
Solutions for Section C (Mechanics 1 and 2)

C1.

$$\mathbf{r}(t) = (2t^2 - t)\mathbf{i} - (3t + 1)\mathbf{j}$$

$$\Rightarrow \mathbf{v}(t) = (4t - 1)\mathbf{i} - 3\mathbf{j} \qquad \mathbf{1}$$

$$\Rightarrow |\mathbf{v}(t)| = \sqrt{(4t - 1)^2 + 9} \qquad \mathbf{1}$$

When the speed is 5,

$$(4t - 1)^2 + 9 = 25 \qquad \mathbf{1}$$

$$(4t - 1)^2 = 16$$

$$4t - 1 = \pm 4$$

$$t = \frac{5}{4} \text{ seconds (as } t > 0). \qquad \mathbf{1}$$

C2. (a)

$$\mathbf{v}_F = 25\sqrt{2}(\cos 45°\mathbf{i} + \sin 45°\mathbf{j}) \qquad \mathbf{1}$$

$$= 25(\mathbf{i} + \mathbf{j})$$

$$\mathbf{r}_F = 25t(\mathbf{i} + \mathbf{j}) \quad \text{as } \mathbf{r}_F(0) = \mathbf{0} \qquad \mathbf{1}$$

$$\mathbf{v}_L = 20\mathbf{j}$$

$$\mathbf{r}_L = 20t\mathbf{j} + \mathbf{c}$$

But $\mathbf{r}_L(0) = 10\mathbf{i}$ so $\mathbf{r}_L = 10\mathbf{i} + 20t\mathbf{j}$ **1**

The position of the ferry relative to the freighter is

$$\mathbf{r}_F - \mathbf{r}_L = (25t - 10)\mathbf{i} + 5t\mathbf{j} \qquad \mathbf{1}$$

(b) When $t = 1$

$$|\mathbf{r}_F - \mathbf{r}_L| = \sqrt{15^2 + 5^2} \qquad \mathbf{1}$$

$$= \sqrt{250} = 5\sqrt{10} \text{ km} \qquad \mathbf{1}$$

C3. (a) Using $T = \frac{2\pi}{\omega} \Rightarrow 8\pi = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{1}{4}$. **1**

Maximum acceleration = $\omega^2 a$ **1**

$$\tfrac{1}{4} = \tfrac{1}{16}a \Rightarrow a = 4 \qquad \mathbf{1}$$

(b) Maximum speed = $\omega a = \frac{1}{4} \times 4 = 1$. **1**

Using

$$v^2 = \omega^2(a^2 - x^2) \qquad \mathbf{1}$$

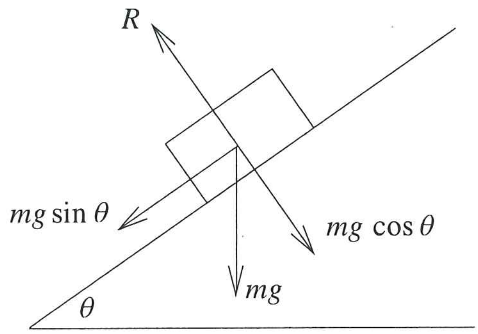
$$\left(\frac{1}{2}\right)^2 = \frac{1}{16}(16 - x^2) \qquad \mathbf{1}$$

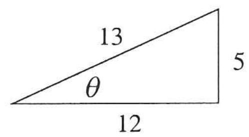
$$4 = 16 - x^2$$

$$x^2 = 12$$

$$x = \pm 2\sqrt{3} \text{ m} \qquad \mathbf{1}$$

C4.





Resolving perp. to plane: $R = mg\cos\theta$

Parallel to the plane (by Newton II)

$$ma = -\mu R - mg\sin\theta$$

$$= -\mu mg\cos\theta - mg\sin\theta \qquad \textbf{2E1}$$

$$a = -g(\mu\cos\theta + \sin\theta)$$

$$= \frac{-(5 + 12\mu)g}{13} \qquad \textbf{2E1}$$

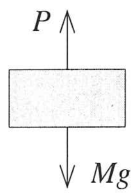
Using $v^2 = u^2 + 2as$

$$0 = gL - \frac{2(5 + 12\mu)gL}{13} \qquad \textbf{1}$$

$$gL = \frac{2(5 + 12\mu)gL}{13}$$

$$10 + 24\mu = 13 \Rightarrow \mu = \frac{1}{8} \qquad \textbf{2E1}$$

C5.



$$P = Mg \qquad \textbf{1}$$

Combined mass = $M + 0{\cdot}01M = 1.01M$.

By Newton II

$$1{\cdot}01Ma = (P + 0{\cdot}05P) - 1{\cdot}01Mg \qquad \textbf{1M,1}$$

$$1.01Ma = 1.05Mg - 1.01Mg$$

$$1.01a = 0.04g \qquad \textbf{1}$$

$$a = \frac{4}{101}g\ (\approx 0.3)\ \text{m s}^{-2} \qquad \textbf{1}$$

C6. (i) By conservation of energy, the speed of block A (v_A) immediately before the collision is given by

$$v_A = \sqrt{2gh}.$$ **1**

By conservation of momentum, the speed of the composite block (v_C) after the collision is given by

$$2mv_C = mv_A$$
$$v_C = \tfrac{1}{2}\sqrt{2gh}$$ **1M,1**

(ii) By the work/energy principle

Work done against friction = Loss of KE + Change in PE **1**

$$F \times h = \tfrac{1}{2}(2m).\tfrac{1}{4}.2gh + 2mg \times \tfrac{1}{2}h$$ **1,1**

$$F = \frac{mg}{2} + mg$$

$$F = \frac{3}{2}W \text{ since } W = mg.$$ **1**

C7. (a) The equations of motion give

$$\ddot{y} = -g \qquad \mathbf{v}(0) = V\cos\alpha\mathbf{i} + V\sin\alpha\mathbf{j}$$
$$\dot{y} = -gt + V\sin\alpha$$
$$y = V\sin\alpha t - \tfrac{1}{2}gt^2$$ **1**

Maximum height when $\dot{y} = 0 \Rightarrow t = \dfrac{V}{g}\sin\alpha$, and so **1**

$$H = V\sin\alpha \times \frac{V}{g}\sin\alpha - \frac{1}{2}g\frac{V^2}{g^2}\sin^2\alpha$$ **1**

$$= \frac{V^2}{2g}\sin^2\alpha$$

(b) (i)

$$h = \frac{V^2}{2g}\sin^2 2\alpha$$

$$= \frac{V^2}{2g}4\sin^2\alpha\cos^2\alpha$$ **1**

$$= \frac{2V^2}{g}\sin^2\alpha(1 - \sin^2\alpha)$$ **1**

$$= 4H\left(1 - \frac{2gH}{V^2}\right) \qquad \text{since } \sin^2\alpha = \frac{2gH}{V^2}$$ **1**

(ii) Since $h = 3H$

$$3H = 4H(1 - \sin^2\alpha)$$ **1**

$$\frac{3}{4} = 1 - \sin^2\alpha$$

$$\sin^2\alpha = \frac{1}{4}$$ **1**

$$\sin\alpha = \pm\frac{1}{2}$$ **1**

$$\Rightarrow \alpha = \frac{\pi}{6} \text{ and so } 2\alpha = \frac{\pi}{3}$$ **1**

C8. (a) Radius of horizontal circle $r = L\sin 60° = \frac{\sqrt{3}}{2}L.$ **1**

$$AB = \frac{r}{\sin 30°} = 2 \times \frac{\sqrt{3}}{2}L = \sqrt{3}\,L$$

Extension of AB, $x = (\sqrt{3} - 1)L$ **1**

Tension in AB, $T_1 = \frac{\lambda x}{L}$ **1**

$= 2(\sqrt{3} - 1)mg.$ **1**

(b) Resolving vertically (where T_2 is the tension in BC)

$$T_1 \cos 30° = mg + T_2 \cos 60°$$ **1**

$$\frac{\sqrt{3}}{2} \times 2(\sqrt{3} - 1)mg = mg + \frac{1}{2}T_2$$ **1**

$$T_2 = (6 - 2\sqrt{3} - 2)mg$$

$$= 2(2 - \sqrt{3})mg$$ **1**

(c) Resolving horizontally (using $L = 1$)

$$T_1 \sin 30° + T_2 \sin 60° = m\left(\frac{\sqrt{3}}{2}\right)\omega^2$$ **1**

$$\frac{1}{2} \times 2(\sqrt{3} - 1)mg + \frac{\sqrt{3}}{2} \times 2(2 - \sqrt{3})mg = m\left(\frac{\sqrt{3}}{2}\right)\omega^2$$ **1**

$$(2\sqrt{3} - 2 + 4\sqrt{3} - 6)g = \sqrt{3}\omega^2$$

$$(6\sqrt{3} - 8)g = \sqrt{3}\,\omega^2$$ **1**

$$\omega^2 = \frac{2(3\sqrt{3} - 4)g}{\sqrt{3}}$$

$$\omega = \sqrt{\frac{2(3\sqrt{3} - 4)g}{\sqrt{3}}}$$ **1**

C9. (i)

$$m\frac{dv}{dt} = -mkv^3 \qquad \mathbf{1}$$

$$v\frac{dv}{dx} = -kv^3 \qquad \mathbf{1}$$

$$\frac{dv}{dx} = -kv^2$$

Separating the variables and integrating gives

$$\int v^{-2}dv = \int -k\,dx \qquad \mathbf{1}$$

$$\Rightarrow -v^{-1} = -kx + c \qquad \mathbf{1}$$

At $x = 0, v = U$

$$-U^{-1} = c$$

so

$$v^{-1} = kx + U^{-1} \qquad \mathbf{1}$$

$$v = \frac{U}{1 + kUx}.$$

(ii) Now $v = \dfrac{dx}{dt}$, so

$$\frac{dx}{dt} = \frac{U}{1 + kUx} \qquad \mathbf{1}$$

$$\int (1 + kUx)\,dx = \int U\,dt \qquad \mathbf{1}$$

$$x + \tfrac{1}{2}kUx^2 = Ut + c_1$$

Since $x = 0$ when $t = 0$, then $c_1 = 0$ **1**

$$kUx^2 + 2x = 2Ut$$

(iii)

$$V = \tfrac{1}{2}U \Rightarrow \tfrac{1}{2}U(1 + kUx) = U$$

$$\Rightarrow 1 + kUx = 2 \Rightarrow x = \frac{1}{kU} \qquad \mathbf{1M,1}$$

The time taken

$$2Ut = kU\frac{1}{k^2U^2} + \frac{2}{kU} = \frac{3}{kU}$$

$$\Rightarrow t = \frac{3}{2kU^2} \qquad \mathbf{1}$$

Advanced Higher Applied Mathematics 2004
Solutions for Section G (Mechanics 1)

G1.

$$\mathbf{r}(t) = (2t^2 - t)\mathbf{i} - (3t + 1)\mathbf{j}$$

$$\Rightarrow \mathbf{v}(t) = (4t - 1)\mathbf{i} - 3\mathbf{j} \qquad \mathbf{1}$$

$$\Rightarrow |\mathbf{v}(t)| = \sqrt{(4t - 1)^2 + 9} \qquad \mathbf{1}$$

When the speed is 5,

$$(4t - 1)^2 + 9 = 25 \qquad \mathbf{1}$$

$$(4t - 1)^2 = 16$$

$$4t - 1 = \pm 4$$

$$t = \frac{5}{4} \text{ seconds (as } t > 0). \qquad \mathbf{1}$$

G2. (a)

$$\mathbf{v}_F = 25\sqrt{2}(\cos 45°\mathbf{i} + \sin 45°\mathbf{j}) \qquad \mathbf{1}$$

$$= 25(\mathbf{i} + \mathbf{j})$$

$$\mathbf{r}_F = 25t(\mathbf{i} + \mathbf{j}) \quad \text{as } \mathbf{r}_F(0) = \mathbf{0} \qquad \mathbf{1}$$

$$\mathbf{v}_L = 20\mathbf{j}$$

$$\mathbf{r}_L = 20t\mathbf{j} + \mathbf{c}$$

But $\mathbf{r}_L(0) = 10\mathbf{i}$ so $\mathbf{r}_L = 10\mathbf{i} + 20t\mathbf{j}$ **1**

The position of the ferry relative to the freighter is

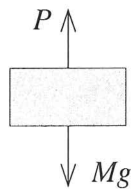
$$\mathbf{r}_F - \mathbf{r}_L = (25t - 10)\mathbf{i} + 5t\mathbf{j} \qquad \mathbf{1}$$

(b) When $t = 1$

$$|\mathbf{r}_F - \mathbf{r}_L| = \sqrt{15^2 + 5^2} \qquad \mathbf{1}$$

$$= \sqrt{250} = 5\sqrt{10} \text{ km} \qquad \mathbf{1}$$

G3.



$$P = Mg \qquad \mathbf{1}$$

Combined mass = $M + 0{\cdot}01M = 1.01M$.

By Newton II

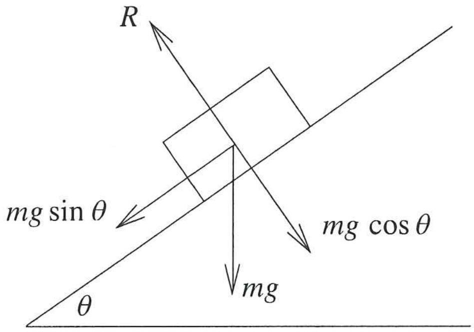
$$1{\cdot}01Ma = (P + 0{\cdot}05P) - 1{\cdot}01Mg \qquad \mathbf{1M,1}$$

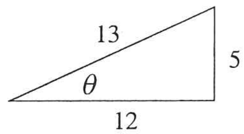
$$1.01Ma = 1.05Mg - 1.01Mg$$

$$1.01a = 0.04g \qquad \mathbf{1}$$

$$a = \frac{4}{101}g\ (\approx 0.3)\text{ m s}^{-2} \qquad \mathbf{1}$$

G4.





Resolving perp. to plane: $R = mg\cos\theta$
Parallel to the plane (by Newton II)

$$ma = -\mu R - mg\sin\theta$$

$$= -\mu mg\cos\theta - mg\sin\theta \qquad \textbf{2E1}$$

$$a = -g(\mu\cos\theta + \sin\theta)$$

$$= \frac{-(5 + 12\mu)g}{13} \qquad \textbf{2E1}$$

Using $v^2 = u^2 + 2as$

$$0 = gL - \frac{2(5 + 12\mu)gL}{13} \qquad \textbf{1}$$

$$gL = \frac{2(5 + 12\mu)gL}{13}$$

$$10 + 24\mu = 13 \Rightarrow \mu = \frac{1}{8} \qquad \textbf{2E1}$$

G5. (a) The equations of motion give

$$\ddot{y} = -g \qquad \mathbf{v}(0) = V\cos\alpha\mathbf{i} + V\sin\alpha\mathbf{j}$$

$$\dot{y} = -gt + V\sin\alpha$$

$$y = V\sin\alpha t - \frac{1}{2}gt^2 \qquad \mathbf{1}$$

Maximum height when $\dot{y} = 0 \Rightarrow t = \frac{V}{g}\sin\alpha$, and so **1**

$$H = V\sin\alpha \times \frac{V}{g}\sin\alpha - \frac{1}{2}g\frac{V^2}{g^2}\sin^2\alpha \qquad \mathbf{1}$$

$$= \frac{V^2}{2g}\sin^2\alpha$$

(b) (i)

$$h = \frac{V^2}{2g}\sin^2 2\alpha$$

$$= \frac{V^2}{2g}4\sin^2\alpha\cos^2\alpha \qquad \mathbf{1}$$

$$= \frac{2V^2}{g}\sin^2\alpha(1 - \sin^2\alpha) \qquad \mathbf{1}$$

$$= 4H\left(1 - \frac{2gH}{V^2}\right) \qquad \text{since} \quad \sin^2\alpha = \frac{2gH}{V^2} \qquad \mathbf{1}$$

(ii) Since $h = 3H$

$$3H = 4H(1 - \sin^2\alpha) \qquad \mathbf{1}$$

$$\frac{3}{4} = 1 - \sin^2\alpha$$

$$\sin^2\alpha = \frac{1}{4} \qquad \mathbf{1}$$

$$\sin\alpha = \pm\frac{1}{2} \qquad \mathbf{1}$$

$$\Rightarrow \alpha = \frac{\pi}{6} \text{ and so } 2\alpha = \frac{\pi}{3} \qquad \mathbf{1}$$