SCOTTISH
QUALIFICATIONS
AUTHORITY

2006 Applied Mathematics

Advanced Higher – Mechanics

Finalised Marking Instructions

General Marking Principles

These principles describe the approach taken when marking Advanced Higher Mathematics papers. For more detailed guidance please refer to the detailed Marking Instructions.

1 The main principle is to give credit for the skills demonstrated and the criteria met. Failure to have a correct method may not preclude a candidate gaining credit for their solution.

2 The answer to one part of a question, even if incorrect, is accepted as a basis for subsequent dependent parts of the question.

3 The following are not penalised:

- working subsequent to a correct answer (unless it provides firm evidence that the requirements of the question have not been met)
- legitimate variation in numerical values / algebraic expressions.

4 Full credit will only be given where the solution contains appropriate working. Where the correct answer might be obtained by inspection or mentally, credit may be given.

5 Sometimes the method to be used in a particular question is explicitly stated; no credit will be given where a candidate obtains the correct answer by an alternative method.

6 Where the method to be used in a particular question is not explicitly stated, full credit will be given for an alternative valid method.

In the detailed Marking Instructions which follow, marks are shown alongside the line for which they are awarded. There are two codes used, M and E. M indicates a method mark, so in question 1, 1M, 1, 1 means a method mark for the product rule (and then a mark for each of the terms). E is shorthand for error. So for example, 2E1, means that a correct answer is awarded 2 marks but that 1 mark is deducted for each error.

Advanced Higher Applied Mathematics 2006
Section A – Mechanics

A1.

$$\mathbf{r}(t) = \left(\frac{1}{3}t^3 - 4t^2\right)\mathbf{i} - \left(2t^2 - 1\right)\mathbf{j}$$

$$\Rightarrow \mathbf{v}(t) = \left(t^2 - 8t\right)\mathbf{i} - 4t\mathbf{j} \qquad \mathbf{1}$$

$$\Rightarrow \mathbf{a}(t) = (2t - 8)\mathbf{i} - 4\mathbf{j} \qquad \mathbf{1}$$

If **a** is in the **j** direction then $t = 4$ so **1**

$$\mathbf{v}(4) = -16\mathbf{i} - 16\mathbf{j} \qquad \mathbf{1}$$

$$|\mathbf{v}(4)| = \sqrt{16^2 + 16^2} = 16\sqrt{2} \qquad \mathbf{1}$$

A2. When $v = v_{\max}$, $x = 0$, so

$$v_{\max} = \omega a = \frac{1}{4}\omega \qquad \mathbf{1}$$

Using

$$v^2 = \omega^2\left(a^2 - x^2\right) \qquad \text{with} \qquad a = \frac{1}{4}$$

$$\text{gives} \qquad v^2 = \omega^2\left(\frac{1}{16} - x^2\right). \qquad \mathbf{1}$$

But $v = \frac{1}{2}v_{\max} = \frac{1}{8}\omega$ so

$$\frac{\omega^2}{64} = \omega^2\left(\frac{1}{16} - x^2\right) \qquad \mathbf{1}$$

$$x^2 = \frac{1}{16} - \frac{1}{64} = \frac{3}{64} \qquad \mathbf{1}$$

$$x = \frac{\pm\sqrt{3}}{8}$$

i.e. the distance from O is $\frac{\sqrt{3}}{8}$ metres. **1**

A3. (a) $\mathbf{a}_L = \frac{1}{8}g\mathbf{j}$, $\mathbf{v}_L(0) = \mathbf{0}$, $\mathbf{r}_L(0) = \mathbf{0}$

$$\Rightarrow \mathbf{v}_L(t) = \frac{gt}{8}\mathbf{j}$$

$$\Rightarrow \mathbf{r}_L(t) = \frac{gt^2}{16}\mathbf{j} \qquad \mathbf{1}$$

Also

$$\mathbf{a}_B = -g\mathbf{j}, \qquad \mathbf{v}_B(0) = \mathbf{0}, \qquad \mathbf{r}_B(0) = 2\mathbf{j}$$

$$\Rightarrow \mathbf{v}_B(t) = -gt\mathbf{j}$$

$$\Rightarrow \mathbf{r}_B(t) = \left(2 - \frac{1}{2}gt^2\right)\mathbf{j} \qquad \mathbf{1}$$

$${}_B\mathbf{r}_L = \left[\frac{-gt^2}{16} + 2 - \frac{gt^2}{2}\right]\mathbf{j} \qquad \mathbf{1}$$

$$= \left[\frac{-9gt^2}{16} + 2\right]\mathbf{j}$$

(b) When ${}_B\mathbf{r}_L = \mathbf{0}$, $gt^2 = \frac{32}{9}$ so **1**

$$\mathbf{r}_B = \left(2 - \frac{1}{2} \times \frac{32}{9}\right)\mathbf{j} = \frac{2}{9}\mathbf{j}$$ **1**

Distance light bulb falls = $2 - \frac{2}{9} = \frac{16}{9}$ metres. **1**

A4. (a) Using $\ddot{x} = 0, \ddot{y} = -g, \mathbf{v} = V\cos\alpha\mathbf{i} + V\sin\alpha\mathbf{j}$

$$x = V\cos\alpha\, t \qquad y = V\sin\alpha\, t - \frac{1}{2}gt^2$$ **1,1**

When $y = 0$, $\quad t = \frac{2V}{g}\sin\alpha \qquad (t > 0)$ **1**

Range, $\quad R = V\cos\alpha \times \frac{2V}{g}\sin\alpha$

$$= \frac{V^2}{g}(2\sin\alpha\cos\alpha)$$ **1**

$$= \frac{V^2\sin 2\alpha}{g}.$$

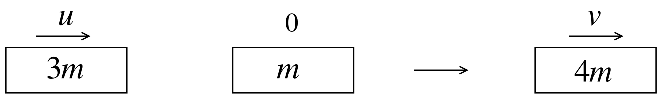
(b) With $\alpha = 15°$ and $R > L$ and $R < 2L$, **1**

$$R > L \Rightarrow \frac{V^2}{gL}\sin 30° > 1 \Rightarrow \frac{V^2}{gL} > 2 \Rightarrow \frac{V}{\sqrt{gL}} > \sqrt{2}$$ **1**

$$R < 2L \Rightarrow \frac{V^2}{2gL} < 2 \Rightarrow \frac{V^2}{gL} < 4 \Rightarrow \frac{V}{\sqrt{gL}} < 2$$ **1**

i.e. $\quad \sqrt{2} < \frac{V}{\sqrt{gL}} < 2.$

A5. (a)



$P_{\text{before}} = 3mu; \qquad P_{\text{after}} = 4mv$

By conservation of momentum

$$v = \tfrac{3}{4}u$$ **1**

Using $v = u + at$, $0 = \frac{3}{4}u + aT \Rightarrow a = \frac{-3u}{4T}$. **1**

and $-R = -4m \times \frac{3u}{4T}$ **1**

$$\Rightarrow R = \frac{3mu}{T}$$ **1**

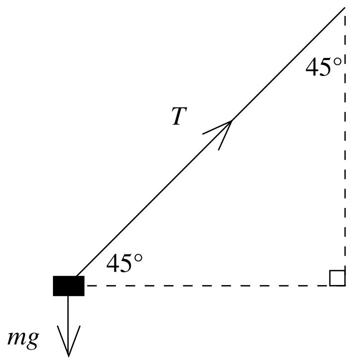
(b) Using $s = ut + \frac{1}{2}at^2$ gives

Distance to rest $= \frac{3}{4}uT - \frac{1}{2}\frac{3u}{4T}T^2$ **1**

$$= \frac{3}{4}uT\left(1 - \frac{1}{2}\right) = \frac{3uT}{8}$$ **1**

Work done $= \frac{3uT}{8} \times \frac{3mu}{T} = \frac{9}{8}mu^2$ Nm **1**

A6.



$$T = \frac{\lambda x}{l} = \frac{8mgx}{l}$$ **1**

(a) Resolving forces vertically

$$\frac{\lambda x}{l}\cos 45° = mg$$ **1**

$$\frac{8mgx}{l\sqrt{2}} = mg$$

$$x = \frac{l}{4\sqrt{2}} = \frac{l\sqrt{2}}{8}$$ **1**

(b) The length of the string is $x + l$.
Resolving horizontally

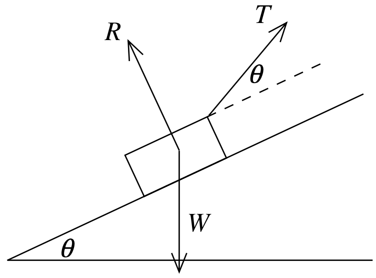
$$m(x + l)\cos 45°\omega^2 = T\sin 45°$$ **1**

$$\frac{m}{\sqrt{2}}\left(1 + \frac{1}{4\sqrt{2}}\right)l\omega^2 = \frac{8mg}{l} \times \frac{l}{4\sqrt{2}} \times \frac{1}{\sqrt{2}}$$ **1**

$$\frac{4\sqrt{2} + 1}{8}l\omega^2 = \frac{8g}{8}$$ **1**

$$\therefore \quad \omega^2 = \frac{8g}{(1 + 4\sqrt{2})\,l}$$

A7. (a)



Resolving perpendicular to the plane

$$W \cos\theta = R + T \sin\theta \qquad (1) \quad \mathbf{1}$$

Resolving parallel to the plane

$$W \sin\theta + \mu R = T \cos\theta \qquad (2) \quad \mathbf{1}$$

From (1)

$$R = W \cos\theta - T \sin\theta.$$

Substituting into (2)

$$W \sin\theta + \mu(W \cos\theta - T \sin\theta) = T \cos\theta \qquad \mathbf{1}$$

$$\Rightarrow \quad T(\cos\theta + \mu \sin\theta) = (\sin\theta + \mu \cos\theta) W \qquad \mathbf{1}$$

$$\Rightarrow \quad T = \frac{\sin\theta + \mu \cos\theta}{\cos\theta + \mu \sin\theta} W \qquad \mathbf{1}$$

$$\Rightarrow \quad T = \frac{\frac{\sin\theta}{\cos\theta} + \mu}{1 + \mu \frac{\sin\theta}{\cos\theta}} W \qquad \mathbf{1}$$

$$\Rightarrow \quad T = \frac{\tan\theta + \mu}{1 + \mu \tan\theta} W.$$

(b) We require $T < W$

$$\text{i.e.} \left(\frac{\tan\theta + \mu}{1 + \mu \tan\theta}\right) W < W \qquad \mathbf{1}$$

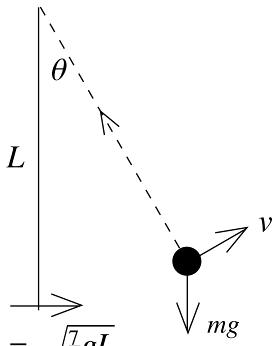
$$\tan\theta + \mu < 1 + \mu \tan\theta \qquad \mathbf{1}$$

$$\tan\theta (1 - \mu) < 1 - \mu$$

$$\tan\theta < 1 \text{ since } 0 < \mu < 1 \text{ and } 0 < \theta < \frac{\pi}{2} \qquad \mathbf{1}$$

$$\text{Thus} \quad 0 < \theta < \frac{\pi}{4}. \qquad \mathbf{1}$$

A8.



$u = \sqrt{\tfrac{7}{2}gL}$

(a) (i) By energy conservation **1**

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgL(1 - \cos\theta) \quad \mathbf{1}$$

$$\frac{1}{2}.\frac{7}{2}gL = \frac{1}{2}v^2 + gL(1 - \cos\theta)$$

$$\text{so } v^2 = \left(\frac{3}{2} + 2\cos\theta\right)gL \quad (1) \quad \mathbf{1}$$

When $\theta = 45°$

$$v^2 = \frac{(3 + 2\sqrt{2})\,gL}{2}$$

$$v = \sqrt{\frac{(3 + 2\sqrt{2})\,gL}{2}} \quad \mathbf{1}$$

(ii) Resolving forces along the line of the string

$$T = \frac{mv^2}{L} + mg\cos\theta \quad \mathbf{1}$$

With $\theta = 45°$

$$T = \frac{m}{L}\frac{(3 + 2\sqrt{2})\,gL}{2} + \frac{mg}{\sqrt{2}} \quad \mathbf{1}$$

$$= \frac{mg}{2}\left\{3 + 2\sqrt{2} + \sqrt{2}\right\}$$

$$= \frac{3}{2}(1 + \sqrt{2})\,mg \quad \mathbf{1}$$

(b) When the string goes slack, $T = 0$ so

$$v^2 = -gL\cos\theta, \quad \mathbf{1}$$

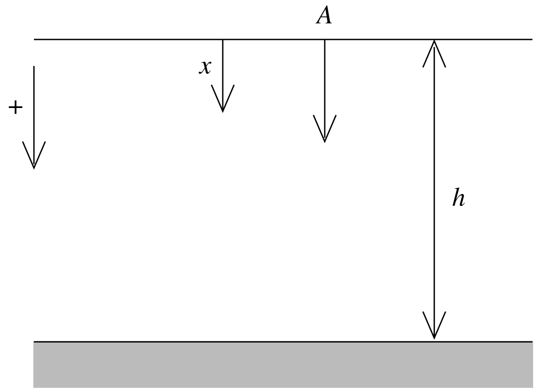
so, in (1)

$$-gL\cos\theta = \left(\frac{3}{2} + 2\cos\theta\right)gL \quad \mathbf{1}$$

$$\Rightarrow \quad 3\cos\theta = -\frac{3}{2} \quad \mathbf{1}$$

$$\Rightarrow \quad \cos\theta = -\frac{1}{2} \quad \Rightarrow \quad \theta = 120° \quad \mathbf{1}$$

A9.



(a)

$$ma = mg - mkv^2 \qquad \mathbf{1}$$

$$v\frac{dv}{dx} = g - kv^2 \qquad \mathbf{1}$$

(b)

$$\int \frac{v\,dv}{g - kv^2} = \int dx \qquad \mathbf{1}$$

Let $w = g - kv^2$

$$dw = -2kv\,dv$$

$$\frac{-1}{2k}\int \frac{dw}{w} = x + c \qquad \mathbf{1}$$

$$\frac{-1}{2k}\ln\left|g - kv^2\right| = x + c \qquad \mathbf{1}$$

As $v = 0$ when $x = 0$, $c = \frac{-1}{2k}\ln g$ **1**

$$x = \frac{1}{2k}\ln g - \frac{1}{2k}\ln\left|g - kv^2\right|$$

$$= \frac{1}{2k}\ln\left|\frac{g}{g - kv^2}\right| \qquad \mathbf{1}$$

$$2kx = \ln\left|\frac{g}{g - kv^2}\right|$$

$$e^{2kx} = \frac{g}{g - kv^2}$$

$$ge^{2kx} - kv^2e^{2kx} = g \qquad \mathbf{1}$$

$$kv^2e^{2kx} = g\left(e^{2kx} - 1\right)$$

$$v^2 = \frac{g}{k}\left(1 - e^{-2kx}\right)$$

(c) When $x = h$ final KE = $\frac{1}{2}mv^2 = \frac{mg}{2k}\left(1 - e^{-2kh}\right) = \frac{mg}{2k}\left(1 - e^{-4}\right)$ **1**

Work done against resistance as fraction of initial PE

$$= \frac{mgh - \frac{mg}{2k}\left(1 - e^{-4}\right)}{mgh} \qquad \mathbf{1}$$

$$= \frac{1 - \frac{1}{4}\left(1 - e^{-4}\right)}{1} \approx 0.755 \qquad \mathbf{1}$$

Section B – Mathematics for Applied Mathematics

B1.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right)$$ **M1, 2E1**

$$\text{So } A^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & 1 \end{pmatrix}.$$

Other valid methods of obtaining A^{-1} will be accepted.

$$\begin{array}{ccccccc} x & + & y & & & = & 1 \\ 2x & + & 3y & + & z & = & 2 \\ 2x & + & 2y & + & z & = & 1 \end{array}$$

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$ **M1,1**

so $x = 0, y = 1, z = -1.$

B2.

$$y = \ln(1 + \sin x)$$

$$\frac{dy}{dx} = \frac{\cos x}{1 + \sin x}$$ **M1,1**

$$\text{so } \frac{d^2y}{dx^2} = \frac{(1 + \sin x)(-\sin x) - \cos x \cos x}{(1 + \sin x)^2}$$ **M1,1**

$$= \frac{-\sin x - 1}{(1 + \sin x)^2}$$ **1**

$$= \frac{-1}{(1 + \sin x)}.$$

B3.

$$S_n = \tfrac{1}{6}n(n + 1)(2n + 1)$$ **1**

$$S_{2n+1} = \tfrac{1}{6}(2n + 1)(2n + 2)(4n + 3)$$ **1**

$$2^2 + 4^2 + \ldots + (2n)^2 = 4(1^2 + 2^2 + \ldots + n^2)$$

$$= \tfrac{2}{3}n(n + 1)(2n + 1)$$ **1**

B4.

$$\cos^2 y \frac{dy}{dx} = y$$

$$\int \frac{dy}{y} = \int \sec^2 x \; dx$$ **M1**

so $\ln y = \tan x + c.$ **1,1**

When $y = 2, x = 0$ giving $c = \ln 2.$ **1**

Hence $\ln y - \ln 2 = \tan x$, i.e. $\ln \tfrac{1}{2}y = \tan x$

$$\Rightarrow y = 2e^{\tan x}.$$ **1**

B5. $1 + x^2 = u \Rightarrow x\,dx = \tfrac{1}{2}\,du$ so **1**

$$\int \frac{x^3}{\sqrt{1 + x^2}}\,dx = \int \frac{(u - 1)}{\sqrt{u}} \frac{1}{2}\,du$$ **1**

$$= \frac{1}{2} \int \left(u^{1/2} - u^{-1/2}\right) du$$ **1**

$$= \frac{1}{3}u^{3/2} - u^{1/2} + c$$ **1**

$$= \frac{1}{3}\left(1 + x^2\right)^{3/2} - \left(1 + x^2\right)^{1/2} + c$$ **1**

$$= \frac{1}{3}\left(x^2 - 2\right)\sqrt{1 + x^2} + c$$

B6. (a)

$$\int_0^1 x\,e^{2x}\,dx = \left[x \int e^{2x}\,dx - \int \tfrac{1}{2}\,e^{2x}\,dx\right]_0^1$$ **M1, 1**

$$= \left[\tfrac{1}{2}\,x\,e^{2x} - \tfrac{1}{4}\,e^{2x}\right]_0^1$$ **1**

$$= \tfrac{1}{2}\,e^2 - \tfrac{1}{4}\,e^2 + \tfrac{1}{4} = \tfrac{1}{4}\left(e^2 + 1\right)$$ **1**

(b)

$$\int_0^1 x^2\,e^{2x}\,dx = \left[x^2 \int e^{2x}\,dx\right]_0^1 - \int_0^1 2x.\tfrac{1}{2}\,e^{2x}\,dx$$ **1**

$$= \left[\tfrac{1}{2}x^2 e^{2x}\right]_0^1 - \int_0^1 x\,e^x\,dx$$ **1**

$$= \left[\tfrac{1}{2}\,e^2 - 0\right] - \tfrac{1}{4}\left(e^2 + 1\right) = \tfrac{1}{4}\left(e^2 - 1\right)$$ **1**

(c)

$$\int_0^1 \left(3x^2 + 2x\right) e^{2x}\,dx = 3\int_0^1 x^2\,e^{2x}\,dx + 2\int_0^1 x\,e^{2x}\,dx$$ **1**

$$= \tfrac{3}{4}\left(e^2 - 1\right) + \tfrac{2}{4}\left(e^2 + 1\right)$$ **1**

$$= \tfrac{1}{4}\left(5e^2 - 1\right)$$

[END OF MARKING INSTRUCTIONS]