X204/701

NATIONAL QUALIFICATIONS 2010

TUESDAY, 25 MAY
1.00 PM – 4.00 PM

APPLIED MATHEMATICS
ADVANCED HIGHER
Mechanics

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer all questions.

   Section A assesses the Units Mechanics 1 and 2
   Section B assesses the Unit Mathematics for Applied Mathematics

3. **Full credit will be given only where the solution contains appropriate working.**

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Section A (Mechanics 1 and 2)

Marks

Answer all the questions.

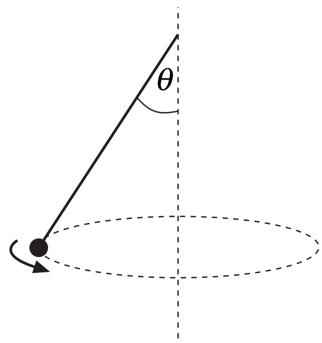
Candidates should observe that g m s^{-2} denotes the magnitude of the acceleration due to gravity.

Where appropriate, take its magnitude to be 9·8 m s^{-2}.

A1. As a set of traffic lights changes to green, a car accelerates uniformly from rest along a straight horizontal road at a m s^{-2}. At the same instant, a lorry travelling at constant speed U m s^{-1} overtakes the car.

Find an expression, in terms of U and a, for the distance travelled by the car when it draws level with the lorry. **4**

A2.



A particle of mass 2 kg is attached to one end of a light inextensible string of length 2 metres. The other end of the string is held fixed while the mass moves in a horizontal circle about a vertical axis at 5 radians per second.

Calculate the size of angle θ, between the string and the vertical axis. **5**

A3. A sledge is released from rest at the top of a ski run which is to be modelled as a rough plane inclined at angle θ to the horizontal. The coefficient of friction between the sledge and ski run surface is μ.

Show that the distance, s metres, travelled down the plane by the sledge to achieve a speed of V m s^{-1} is given by

$$s = \frac{V^2}{2g(\sin\theta - \mu\cos\theta)}.$$ **4**

A4. Abi is cycling along a horizontal road at 4 m s^{-1}. The combined mass of Abi and her cycle is 100 kg. She applies the brakes, exerting a braking force of magnitude $8t$ newtons, where t seconds is the time from the instant the brakes are applied.

Calculate:

(a) the time for the cycle to come to rest; **3**

(b) the stopping distance of the cycle. **2**

Marks

A5. A catapult exerts a force $F(t) = 100\cos\frac{1}{2}\pi t$ newtons on a stone for $0 \le t \le 1$, where t seconds is the time that the stone is in contact with the catapult.

Calculate the change in momentum of the stone. **3**

A6. A toy car of mass 250 grams is stationary on a smooth horizontal surface. One end of a light spring is attached to the car, the other end is fixed to the surface. The natural length of the spring is 1 metre and the modulus of elasticity is 4 newtons.

The car is pulled along the surface, extending the spring by 20 centimetres, and then released.

(*a*) Show that the displacement, x metres, of the car from its equilibrium position satisfies an equation of the form

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

where the value of the constant ω should be stated. **3**

(*b*) Calculate the maximum speed of the car. **2**

A7. A rocket of mass 100 kg is fired vertically upwards from ground level under constant gravity and the opposing force of air resistance. The magnitude of the air resistance is $25\,v^2$ newtons per unit mass, where v is the speed of the rocket in metres per second.

Show that the maximum height reached by the rocket is given by

$$\frac{1}{50}\ln\left(\frac{g + 25U^2}{g}\right)$$

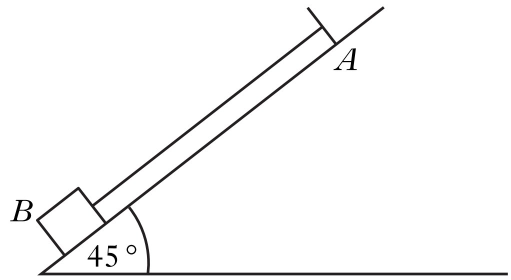
where $U\,\text{m s}^{-1}$ is the initial speed of the rocket. **6**

[It may be assumed that $\int \frac{x}{a^2 + b^2x^2}\,dx = \frac{1}{2b^2}\ln\left|a^2 + b^2x^2\right| + c$, where $a, b > 0$.]

[Turn over

Marks

A8. A block of mass 2 kg is held at rest on a rough slope inclined at an angle of 45° to the horizontal. A light spring has one end fixed at a point A and the other end is attached to the block, B. The natural length of the spring is 1 metre and its elastic modulus is λ newtons. The initial distance between A and B is 2 metres and the coefficient of friction between the block and the plane is μ.

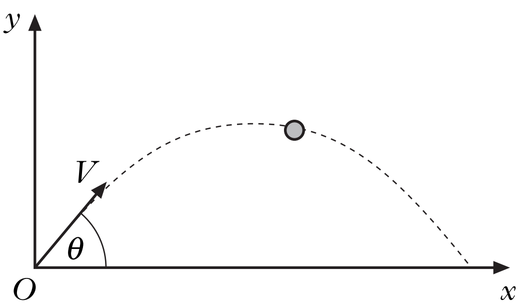


The mass is released and travels 25 centimetres up the slope in a straight line before coming to rest. Show that

$$\lambda = \frac{8\sqrt{2}}{7}(1+\mu)g.$$ **6**

Marks

A9. Bobbie kicks a football from the origin O on a horizontal football pitch. The ball is projected at speed $V\,\mathrm{m\,s^{-1}}$ at an angle θ to the horizontal and moves freely under gravity.

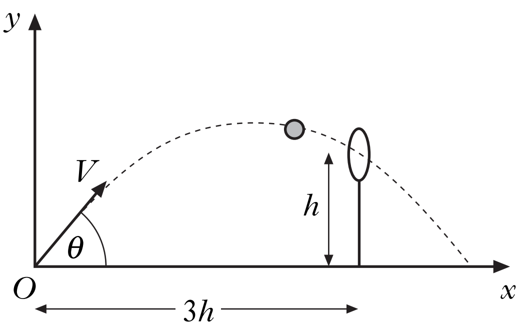


(*a*) Given that Ox and Oy are rectangular axes as indicated in the diagram, show from the equations of motion that the trajectory of the ball is given by

$$y = x\tan\theta - \frac{gx^2}{2V^2}(1+\tan^2\theta).$$ **3**

[Note that $\sec^2\theta = 1 + \tan^2\theta$.]

(*b*) The ball passes through the centre of a hoop with its trajectory unchanged. The centre of the hoop is at $(3h, h)$ and the speed of projection is given by $V = 3\sqrt{\frac{gh}{2}}$.



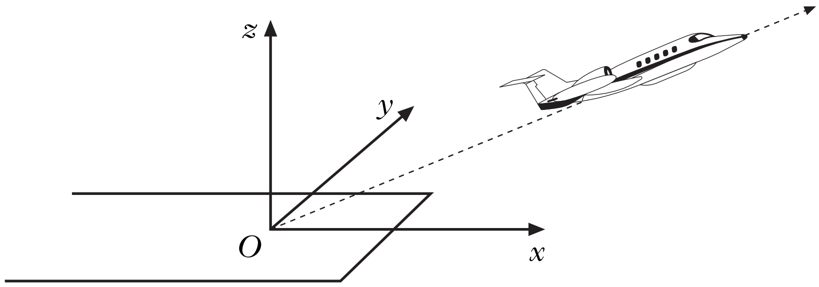
Determine the two possible values of $\tan\theta$. **4**

When $\tan\theta$ takes the larger of these values, find an expression for the range of the football in terms of h. **3**

[Turn over

Marks

A10. Relative to the rectangular coordinate system as shown in the diagram, a horizontal runway is aligned along the Ox direction. The unit vectors, **i**, **j** and **k** are in the x, y and z directions respectively.



Aircraft A takes off from a point O on the runway and thereafter climbs with constant speed $V\,\text{m s}^{-1}$ at an angle of $45°$ to the horizontal in the x-z plane.

(a) Find the position vector of the aircraft A, t seconds after it takes off. **2**

A second aircraft B is travelling with a constant velocity vector

$$\mathbf{v}_B = \frac{V}{\sqrt{2}}\mathbf{j}.$$

At the moment that aircraft A takes off, the position vector of B is

$$\mathbf{r}_B = L(\mathbf{i} - \mathbf{j} + 4\mathbf{k}),$$

where L is a positive constant.

(b) Find the position vector of aircraft B, t seconds after aircraft A has taken off. **2**

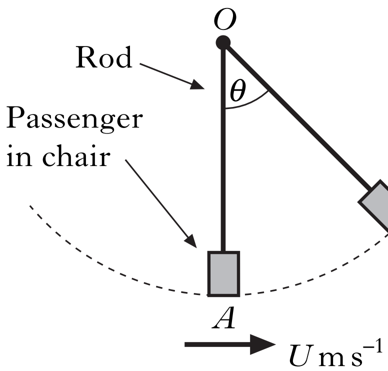
(c) Show that the distance D metres, between the aircraft is given by

$$D^2 = \tfrac{3}{2}V^2t^2 - 6\sqrt{2}VLt + 18L^2.$$ **3**

Hence, find the minimum distance between the aircraft, in terms of the constant L. **3**

Marks

A11. A fairground ride consists of a rod that is free to rotate in the vertical plane about a fixed point O. A passenger sits in a chair that is attached to the other end of the rod, as shown in the diagram.



The ride can be modelled as a particle attached to a light inextensible rod. The angle between the rod and the vertical OA is θ radians, the length of the rod is R metres and the mass of the particle is M kg. The maximum value of the angular displacement θ is denoted by α. The particle is initially at a point A, vertically below the fixed point O.

(*a*) The particle is given an initial speed of $U\,\text{m s}^{-1}$ when it is at A and this causes the particle to oscillate through an arc with $\alpha = \frac{\pi}{4}$.

Show that

$$U^2 = gR(2 - \sqrt{2}).$$ **2**

The amplitude of the oscillations is increased such that $-\pi < \theta < \pi$ and the speed of the particle at A is $u\,\text{m s}^{-1}$.

(*b*) Find an expression for the speed $v\,\text{m s}^{-1}$ of the particle at any point in the oscillation, in terms of u, g, R and θ. **2**

(*c*) Given that the tension in the rod, as a function of angle θ, is denoted by $T(\theta)$ and assuming that $\frac{3\pi}{4} < \alpha < \pi$, show that

$$T\left(\frac{\pi}{4}\right) - T\left(\frac{3\pi}{4}\right) = kMg$$

where k is a constant to be obtained. **6**

[*END OF SECTION A*]

[Turn over for Section B on *Pages eight* and *nine*

Section B (Mathematics for Applied Mathematics)

Marks

Answer all the questions.

B1. Differentiate the following, simplifying your answers as appropriate.

(*a*) $f(x) = e^{2x}\tan x,\ \frac{-\pi}{2} < x < \frac{\pi}{2}.$ **3**

(*b*) $g(x) = \dfrac{\cos 2x}{x^3}.$ **4**

B2. Find the term in a^6 in the binomial expansion of $\left(\dfrac{1}{a} + 3a\right)^{10}.$ **4**

B3. Express $\dfrac{3x}{(x+1)^2}$ in partial fractions. **3**

Hence obtain $\displaystyle\int \frac{3x}{(x+1)^2}\,dx$ **2**

B4. An industrial process is modelled by the differential equation

$$\frac{dy}{dt} = \frac{9te^{3t}}{y},$$

where $y > 0$ and $t \geq 0$.

Given that $y = 2$ when $t = 0$, find y explicitly in terms of t. **7**

Marks

B5. (*a*) Find the value(s) of m for which the matrix

$$A = \begin{pmatrix} m & 1 & 1 \\ 0 & m & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

is singular. **3**

(*b*) The matrix $B = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & -3 \end{pmatrix}$. Use elementary row operations to obtain B^{-1}. **4**

Hence, or otherwise, solve the system of equations

$$\begin{aligned} x + y - z &= 3 \\ y + z &= -2 \\ x - 3z &= 7. \end{aligned}$$

2

[*END OF SECTION B*]

[*END OF QUESTION PAPER*]

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