2011 Applied Mathematics

Advanced Higher – Mechanics

Finalised Marking Instructions

General Marking Principles

These principles describe the approach taken when marking Advanced Higher Applied Mathematics papers. For more detailed guidance please refer to the detailed Marking Instructions.

1 The main principle is to give credit for the skills demonstrated and the criteria met. Failure to have a correct method may not preclude a candidate gaining credit for their solution.

2 The answer to one part of a question, even if incorrect, can be accepted as a basis for subsequent dependent parts of the question.

3 The following are not penalised:

- working subsequent to a correct answer (unless it provides firm evidence that the requirements of the question have not been met)
- legitimate variation in numerical values / algebraic expressions.

4 Full credit will only be given where the solution contains appropriate working. Where the correct answer might be obtained by inspection or mentally, credit may be given.

5 Sometimes the method to be used in a particular question is explicitly stated; no credit will be given where a candidate obtains the correct answer by an alternative method.

6 Where the method to be used in a particular question is not explicitly stated in the question paper, full credit is available for an alternative valid method. (Some likely alternatives are included but these should not be assumed to be the only acceptable ones.)

In the detailed Marking Instructions which follow, marks are shown alongside the line for which they are awarded. There are two codes used M and E. The code M indicates a method mark, so in question B1(a), 1M means a method mark for the product rule.

Advanced Higher Applied Mathematics 2011

Mechanics Solutions

A1. Let **v** be the velocity of the 2 kg mass before collision.
By conservation of momentum

$$2\mathbf{v} + (-2\mathbf{i} + 4\mathbf{j}) = 3(\mathbf{i} + 4\mathbf{j}) \qquad \mathbf{1}$$

$$\Rightarrow \mathbf{v} = 2\tfrac{1}{2}\mathbf{i} + 4\mathbf{j} \qquad \mathbf{1}$$

$$\text{Hence } |\mathbf{v}| = \sqrt{\frac{25}{4} + 16}\left(= \sqrt{\frac{89}{4}} = \frac{1}{2}\sqrt{89}\right) \qquad \mathbf{1}$$

$$\approx 4{\cdot}72 \text{ ms}^{-1}$$

A2. (a) Let the time when the car passes B be t_B seconds. Then

$$v_1(t_B) = v_2(t_B)$$

$$\Rightarrow -\frac{1}{2}t_B + 45 = \frac{1}{8}t_B + \frac{15}{2} \qquad \mathbf{1M}$$

$$\Rightarrow 5t_B = 300$$

$$t_B = 60 \qquad \mathbf{1}$$

$$\Rightarrow v_1(t_B) = -\frac{1}{2} \times 60 + 45.$$

The speed of the car at B is 15 ms^{-1}. **1**

(b)

$$v_C = v_2(100)$$

$$= \frac{1}{8} \times 100 + \frac{15}{2}$$

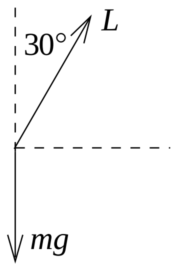
$$= 20 \qquad \mathbf{1}$$

$$\text{Distance travelled} = \text{area under the graph} \qquad \mathbf{1M}$$

$$= 15 \times 40 + 0{\cdot}5 \times 5 \times 40 + 0{\cdot}5 \times 20 \times 20$$

$$= 900 \text{ metres} \qquad \mathbf{1}$$

A3.



1

Resolving vertically

$$L\ \cos 30° = mg \qquad \mathbf{1}$$

$$\Rightarrow L = \frac{2}{\sqrt{3}} mg. \qquad (*)$$

Resolving horizontally

$$\frac{mU^2}{r} = L\ \sin 30° \qquad \mathbf{1}$$

$$\Rightarrow L = \frac{2mU^2}{R} \qquad (**)$$

Eliminating L between (*) and (**) gives

$$\frac{U^2}{R} = \frac{g}{\sqrt{3}} \qquad \mathbf{1}$$

$$\Rightarrow R = \frac{\sqrt{3}U^2}{g}.$$

The orbital period is

$$T = \frac{2\pi R}{U} \qquad \mathbf{1}$$

and using the expression for R

$$T = \frac{2\sqrt{3}\pi U}{g}. \qquad \mathbf{1}$$

A4. Let t hours be the time measured from 09:00. For the passenger plane

$$\mathbf{v}_P = \begin{pmatrix} 300 \\ 400 \end{pmatrix}, \qquad \mathbf{r}_P(0) = \begin{pmatrix} -100 \\ 250 \end{pmatrix}$$

and so

$$\mathbf{r}_P(t) = \begin{pmatrix} -100 \\ 250 \end{pmatrix} + \begin{pmatrix} 300 \\ 400 \end{pmatrix} t. \qquad \mathbf{1}$$

$$= \begin{pmatrix} -100 + 300t \\ 250 + 400t \end{pmatrix}.$$

For the military aircraft

$$\mathbf{v}_M = \begin{pmatrix} 600 \\ 500 \end{pmatrix}, \qquad \mathbf{r}_M(0{\cdot}5) = \begin{pmatrix} -100 \\ 400 \end{pmatrix}.$$

Integrating the velocity gives

$$\mathbf{r}_M(t) = \begin{pmatrix} 600 \\ 500 \end{pmatrix} t + \mathbf{c}. \qquad \mathbf{1}$$

And applying the condition on the position of the military plane at $t = 0{\cdot}5$ gives

$$\mathbf{c} = \begin{pmatrix} -400 \\ 150 \end{pmatrix}$$

$$\Rightarrow \mathbf{r}_M(t) = \begin{pmatrix} 600 \\ 500 \end{pmatrix} t + \begin{pmatrix} -400 \\ 150 \end{pmatrix} \qquad \mathbf{1}$$

$$= \begin{pmatrix} 600t - 400 \\ 500t + 150 \end{pmatrix}.$$

For the aircraft to be on collision course there has to be a positive solution to

$$\mathbf{r}_M(t) = \mathbf{r}_P(t). \qquad \mathbf{1M}$$

From the x-components

$$600t - 400 = 300t - 100 \Rightarrow t = 1,$$

and the y-components give

$$400t + 250 = 500t + 150 \Rightarrow t = 1. \qquad \mathbf{1}$$

Hence the aircraft are on a collision course.

A5. Starting from

$$v\frac{dv}{dx} = -\omega^2 x$$

and integrating gives

$$\int v\,dv = -\int \omega^2 x\,dx$$ **1M**

$$\text{i.e.} \quad \frac{1}{2}v^2 = -\frac{1}{2}\omega^2 x^2 + c.$$ **1**

Applying the condition $v = 0$ when $x = 0{\cdot}05$ gives

$$2c = 2{\cdot}5 \times 10^{-3}\omega^2$$

so

$$v^2 = \omega^2\left(0{\cdot}0025 - x^2\right).$$ **1**

But

$$\omega = \frac{2\pi}{T}$$

$$\Rightarrow \omega = \frac{2\pi}{1{\cdot}5 \times 10^{-4}}$$ **1**

and hence

$$v_{\text{max}} = \omega \times 0{\cdot}05$$ **1**

$$= \frac{2\pi}{1{\cdot}5 \times 10^{-4}} \times 0{\cdot}05 = \frac{2000\pi}{3}$$

$$\approx 2{\cdot}1 \times 10^3 \text{ ms}^{-1}.$$ **1**

A6.

$$\mathbf{v} = 8\left(1 - e^{-2t}\right)\mathbf{i}$$

So differentiating gives

$$\mathbf{a} = 16e^{-2t}\mathbf{i} \qquad \mathbf{1}$$

and by Newton 2

$$\mathbf{F} = 0{\cdot}25 \times 16e^{-2t}\mathbf{i}$$

$$= 4e^{-2t}\mathbf{i} \text{ newtons}. \qquad \mathbf{1}$$

The work done is given by

$$W = \int_0^1 \mathbf{F}.\mathbf{v}\,dt \qquad \mathbf{1}$$

$$= \int_0^1 4e^{-2t} \times 8\left(1 - e^{-2t}\right)dt \qquad \mathbf{1}$$

$$= 32\int_0^1 \left(e^{-2t} - e^{-4t}\right)dt$$

$$= 32\left[-\tfrac{1}{2}e^{-2t} + \tfrac{1}{4}e^{4t}\right]_0^1 \qquad \mathbf{1}$$

$$\text{i.e.} \quad W = 8\left(e^{-4} - 2e^{-2} + 1\right) \text{ joules} \qquad \mathbf{1}$$

$$= 8\left(e^{-2} - 1\right)^2 \text{ joules} \left(\approx 5.98 \text{ joules}\right)$$

A7. Let the masses of S, A and B be denoted by m, M, $4M$ and let $\theta = \angle SAB$.
The gravitational attraction between S and A is

$$F_{AS} = \frac{GMm}{L^2}. \qquad \mathbf{1}$$

The gravitational attraction between S and B is

$$F_{BS} = \frac{G(4M)m}{(2L)^2} = \frac{GMm}{L^2}. \qquad \mathbf{1}$$

Then by resolving the forces parallel and perpendicular to AB

$$F_1 = F_{BS}\sin\theta - F_{AS}\cos\theta \qquad \mathbf{1}$$

$$\Rightarrow F_1 = \frac{GMm}{L^2}\left(\sin\theta - \cos\theta\right)$$

$$= \frac{GMm}{L^2}\left(\frac{2}{\sqrt{5}} - \frac{1}{\sqrt{5}}\right) = \frac{GMm}{\sqrt{5}L^2} \qquad \mathbf{1}$$

Similarly

$$F_2 = F_{AS}\sin\theta + F_{BS}\cos\theta \qquad \mathbf{1}$$

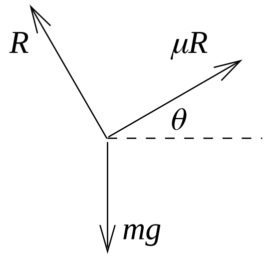
$$\Rightarrow F_2 = \frac{3GMm}{\sqrt{5}L^2}.$$

Hence

$$\frac{F_2}{F_1} = \frac{\dfrac{3GMm}{\sqrt{5}L^2}}{\dfrac{GMm}{\sqrt{5}L^2}} \qquad \mathbf{1}$$

$$\text{i.e.} \quad F_2 = 3F_1$$

A8. *Method 1*



By Newton's second law

$$R = mg\cos\theta$$

$$ma = mg\sin\theta - \mu R \qquad \textbf{1}$$

hence

$$\Rightarrow a = g\sin\theta - \mu g\cos\theta. \qquad \textbf{1}$$

Using the equation $v^2 = u^2 + 2as$,

$$U^2 = 2(4L)g(\sin\theta - \mu\cos\theta) \qquad \textbf{1}$$

and hence

$$\frac{U^2}{gL} = 8(\sin\theta - \mu\cos\theta). \qquad (*)$$

When the box is projected up the plane, Newton's second law gives

$$ma = -mg\sin\theta - \mu mg\cos\theta \qquad \textbf{1}$$

$$\Rightarrow a = -g(\sin\theta + \mu\cos\theta).$$

Again using $v^2 = u^2 + 2as$, we get

$$0 = U^2 + 2(3L)(-g(\sin\theta + \mu\cos\theta)) \qquad \textbf{1}$$

$$\Rightarrow \frac{U^2}{gL} = 6(\sin\theta + \mu\cos\theta). \qquad (**) \qquad \textbf{1}$$

Equating (*) and (**) gives

$$8(\sin\theta - \mu\cos\theta) = 6(\sin\theta + \mu\cos\theta). \qquad \textbf{1}$$

$$\Rightarrow \sin\theta = 7\mu\cos\theta \qquad \textbf{1}$$

and hence

$$\mu = \frac{1}{7}\tan\theta.$$

When $\tan\theta = \frac{3}{4}$, $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$ and $\mu = \frac{3}{28}$. **1**

Hence from (**),

$$\frac{U^2}{gL} = 6\left(\frac{3}{5} + \frac{3}{28} \times \frac{4}{5}\right) = 6 \times \frac{21 + 3}{35} = \frac{144}{35}$$

$$\Rightarrow L = \frac{35U^2}{144g}\left(\approx 0.243\frac{U^2}{g}\right). \qquad \textbf{1}$$

Method 2

Initially: PE $= mgh = 4Lmg\sin\theta$ and KE = 0

Finally: PE = 0 and KE $= \tfrac{1}{2}mU^2$

Energy lost = $4Lmg\sin\theta - \tfrac{1}{2}mU^2$. **1**

Change in energy = Work done against friction **1M**

$$4Lmg\sin\theta - \frac{1}{2}mU^2 = \mu R \times 4L = 4L\mu mg\cos\theta \qquad \mathbf{1}$$

$$4mgL\left(\sin\theta - \mu\cos\theta\right) = \frac{1}{2}mU^2$$

$$\frac{U^2}{gL} = 8\left(\sin\theta - \mu\cos\theta\right) \;\ldots\;\ldots\;(i)$$

At start of upward motion: PE = 0 and KE $= \tfrac{1}{2}mU^2$.

At top of upward motion: PE $= 3mgL\sin\theta$ and KE = 0.

Energy lost = $\tfrac{1}{2}mU^2 - 3mgL\sin\theta$. **1**

Thus

$$\frac{1}{2}mU^2 - 3mgL\sin\theta = \mu R \times 3L = 3\mu Lmg\sin\theta \qquad \mathbf{1}$$

$$U^2 = 6gL\left(\sin\theta + \mu\cos\theta\right)$$

$$\frac{U^2}{gL} = 6\left(\sin\theta + \mu\cos\theta\right)\ldots\;\ldots\;(ii) \qquad \mathbf{1}$$

Equating (i) and (ii)

$$8\sin\theta - 8\mu\cos\theta = 6\sin\theta + 6\mu\cos\theta \qquad \mathbf{1}$$

$$2\sin\theta = 14\mu\cos\theta \qquad \mathbf{1}$$

$$\mu = \frac{1}{7}\tan\theta$$

When $\tan\theta = \dfrac{3}{4}$, $\sin\theta = \dfrac{3}{5}$, $\cos\theta = \dfrac{4}{5}$ and $\mu = \dfrac{3}{28}$. **1**

Hence from (ii),

$$L = \frac{U^2}{6gL\left(\sin\theta + \mu\cos\theta\right)} = \frac{U^2}{6gL\left(\frac{3}{5} + \frac{3}{28} \times \frac{4}{5}\right)}$$

$$= \frac{35U^2}{144g}\left(\approx 0.243\frac{U^2}{g}\right). \qquad \mathbf{1}$$

A9. (a) Let the initial speed of the ball be V ms^{-1} . From the equation of motion for the horizontal component

$$x = Vt\cos\theta \Rightarrow 12 = (30\cos 5°)t \Rightarrow t \approx 0{\cdot}4015. \qquad \textbf{1}$$

The vertical component of motion is governed by

$$y = (V\sin\theta)t + \tfrac{1}{2}gt^2 \qquad \textbf{1}$$

so when the ball passes over the net, the distance it has gone down is given by

$$(30\sin 5°) \times 0{\cdot}4015 + 0{\cdot}5 \times 9{\cdot}8 \times 0{\cdot}4015^2 \approx 1{\cdot}839$$

hence the ball clears the net by $(2 - 1{\cdot}839)$ m $\approx$ 16 cm. **1**

(b) Apply conservation of energy.
Initial KE = $\tfrac{1}{2}m(6gH) = 3mgH$.
PE lost = mgH.
So KE at impact = $4mgH$ **1**

$$\Rightarrow \frac{1}{2}mv^2 = 4mgH$$

The speed of the ball on impact with the court is then

$$\sqrt{8gH} = 2\sqrt{2gH}. \qquad \textbf{1}$$

Considering the horizontal component of the speed, the angle θ satisfies

$$\sqrt{8gH}\cos\theta = \sqrt{6gH} \qquad \textbf{1M}$$

$$\Rightarrow \cos\theta = \sqrt{\frac{6gH}{8gH}} = \frac{\sqrt{3}}{2} \Rightarrow \theta = 30°. \qquad \textbf{1}$$

After impact the kinetic energy of the ball is

$$0{\cdot}5 \times (0{\cdot}5 \times m \times 8gH) = 2mgH. \qquad \textbf{1}$$

By energy conservation, when the ball attains its maximum height,

$$\frac{1}{2}mv^2 = 2mgH - mg \times 0{\cdot}2H. \qquad \textbf{1}$$

Hence

$$v^2 = \frac{18gH}{5}$$

$$\text{i.e.} \quad v = 3\sqrt{\frac{2gH}{5}}. \qquad \textbf{1}$$

Alternative
The start of (b) can easily be done using equations of motion.
The vertical speed is $\sqrt{2gh}$.
The overall speed is $\sqrt{8gh} = 2\sqrt{2gh}$ which leads to
$\sin\theta = \tfrac{1}{2} \Rightarrow \theta = 30°$.

A10. By Newton's second law

$$ma = -mg - mkv^2 \qquad \mathbf{1}$$

and in differential form

$$v\frac{dv}{dx} = -\left(g + kv^2\right). \qquad \mathbf{1}$$

Separating the variables gives

$$\int \frac{v\,dv}{g + kv^2} = -\int dx \qquad \mathbf{1M}$$

$$\text{i.e.} \quad \frac{1}{2k}\int \frac{2kv\,dv}{g + kv^2} = -\int dx$$

and performing the integration

$$\ln\left(g + kv^2\right) = -2kx + C, \qquad \mathbf{1}$$

where C is a constant of integration.
Applying the initial condition $v = U$ when $x = 0$, gives

$$C = \ln\left(g + kU^2\right)$$

and hence

$$\ln\left(g + kv^2\right) = -2kx + \ln\left(g + kU^2\right). \qquad \mathbf{1}$$

Collecting logs and taking exponentials leads to

$$g + kv^2 = \left(g + kU^2\right)e^{-2kx} \qquad \mathbf{1}$$

from which we obtain the required result

$$v^2 = \left(\frac{g}{k} + U^2\right)e^{-2kx} - \frac{g}{k}.$$

As $U = \sqrt{\frac{g}{k}}$,

$$v^2 = U^2\left(2e^{-2kx} - 1\right)$$

and the kinetic energy E_{KE} is

$$E_{\mathrm{KE}} = \tfrac{1}{2}mU^2\left(2e^{-2kx} - 1\right) \qquad (*) \qquad \mathbf{1}$$

The maximum height x_m is attained when $v = 0$, so

$$e^{-2kx_m} = \tfrac{1}{2}. \qquad \mathbf{1}$$

At $x = \tfrac{1}{2}x_m$,

$$e^{-2kx} = e^{-kx_m} = \left(e^{-2kx_m}\right)^{1/2} = \left(\tfrac{1}{2}\right)^{1/2}. \qquad \mathbf{1}$$

The required kinetic energy is

$$E_{\mathrm{KE}} = \frac{1}{2}mU^2\left(\frac{2}{\sqrt{2}} - 1\right) \qquad \mathbf{1}$$

from (*) and the stated result follows.