## 1.2 Applying Algebraic and Calculus skills to Properties of Function

*Revising the definition of a function*

- Know the definition of a function.
- Identify the domain and range of a function including and restrictions.

A function is rule which assigns each member of an input set (the **domain**) to exactly one member of an output set (the co-domain). The subset of the co-domain containing all the outputs is called the **range**.

**Examples**
Write down the largest suitable domain for each function and state the corresponding range.

$f(x) = \sin x$ $\quad x \in \mathbb{R},$
$-1 \leq f(x) \leq 1$

$f(x) = \sqrt{x-2}$ $\quad x \in \mathbb{R}, \quad x \geq 2$
$f(x) \geq 0$

$f(x) = x!$ $\quad x \in \mathbb{W}$
$f(x) = \{1, 2, 6, 24, 120, \ldots\}$

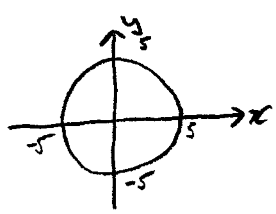
$f(x) \tan x$ $\quad x \in \mathbb{R}, \quad x \neq \frac{n\pi}{2}$
$f(x) \in \mathbb{R}$

$f(x) = x^2$ $\quad x \in \mathbb{R}$
$f(x) \geq 0$

$f(x) = \sqrt{x^2}$ $\quad x \in \mathbb{R}$
$f(x) \geq 0$

$f(x) = \dfrac{1}{\sin x}$ $\quad x \in \mathbb{R}, \quad x \neq n\pi$
$f(x) \leq -1$ or $f(x) \geq 1$

Sketch $x^2 + y^2 = 25$.



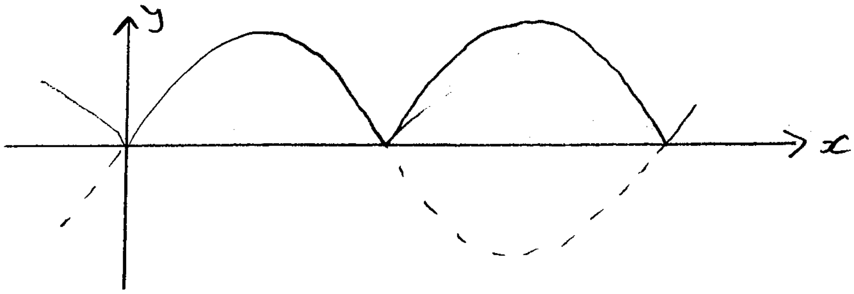

$x^2 + y^2 = 25$ does not define a function on ***R*** because every value of $x$ does not correspond to exactly one value of $y$.
To work with this relationship as a function restrict the domain to $-5 \leq x \leq 5$ range to $y \geq 0$

*Learning to sketch the modulus function*

$|x| = \{x \geq 0$
$\quad\;\; \{-x < 0$

To sketch the modulus function, reflect the negative portion in the $x$-axis.

**Example**
$y = |\sin x|$

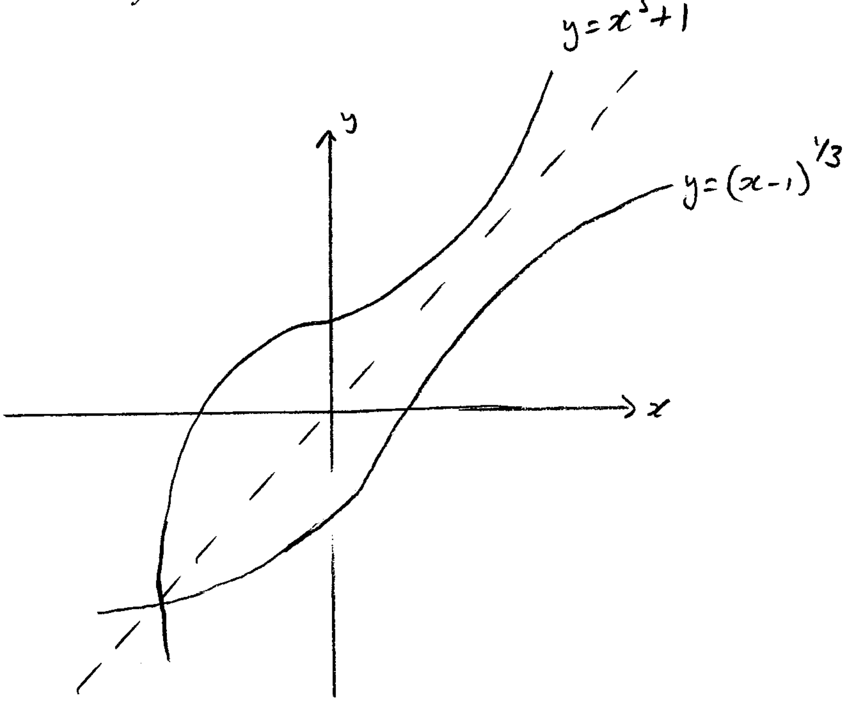



*Revising the inverse function*

A function has an inverse if there is a one-to-one correspondence between the domain and range.

To sketch the inverse, reflect in the line $y = x$.
To find the formula, interchange $x$ and $y$ and make $y$ the subject.

**Example**
$y = x^3 + 1$




$x = y^3 + 1$

$(x-1) = y^3$

$y = (x-1)^{1/3}$

*Learning to find the extrema of functions*

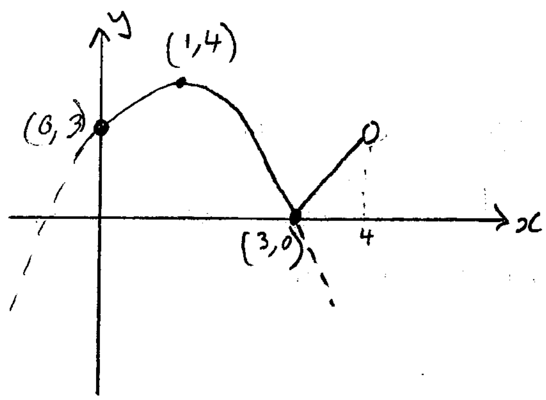
- Understand the definition of a critical point
- Find stationary points and determine their nature
- Examine critical points and identify the global minimum and maximum

A critical point is where $f'(x) = 0$ or $f'(x)$ is undefined.

**Examples**

1. $f(x) = |3 + 2x - x^2|$ defined on the domain $[0,4)$

$= |(3-x)(1+x)|$




(0,3) is an end-point minimum

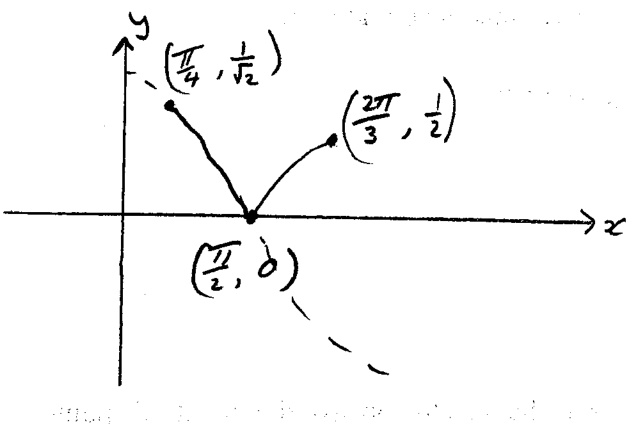
(1,4) is a maximum turning point

(3,0) is a local minimum

At $x=4$ the function is not defined.

The global minimum is (3,0) and the global maximum is (1,4).

2. $f(x) = |\cos x|$ defined on the domain $\left[\frac{\pi}{4}, \frac{2\pi}{3}\right]$




$\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ is a local maximum.

$\left(\frac{\pi}{2}, 0\right)$ is a local minimum

$\left(\frac{2\pi}{3}, \frac{1}{2}\right)$ is a local maximum.

The global minimum is $\left(\frac{\pi}{2}, 0\right)$ and the global maximum is $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$

*Learning to use concavity to determine the nature of stationary points*

- Understand the second derivative as the rate of change of gradient
- Know conditions for minimum and maximum turning point
- Use changes in concavity to prove the existence of points of inflexion

$f'(x)$ is the rate of change of the function - we call this gradient.
$f''(x)$ is the rate of change of the gradient.

Consider a function with the gradient changing from positive to negative:

*i.e.* a maximum turning point. The gradient is decreasing so $f''(x)$ is negative.

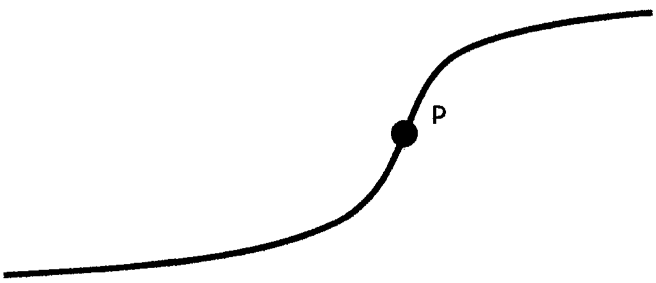
$$\boxed{f''(x) < 0 \Rightarrow max\ turning\ point}$$

Similarly, a function with the gradient changing from negative to positive:

is a minimum turning point. The gradient is increasing so $f''(x)$ is positive.

$$\boxed{f''(x) > 0 \Rightarrow min\ turning\ point}$$

In line with the shape of the curve, a region where the gradient is decreasing is said to be **concave down** and a region where the gradient is increasing is **concave up**.




Initially the curve is concave up. At point P the curve changes to concave down. At this point the gradient is neither increasing nor decreasing, $f''(x) = 0$.
A point where the concavity changes is called a **point of inflexion.**
If $f''(x) = 0$ or $f''(x)$ does not exist there may be a point of inflexion. A table of signs for $f''(x)$ can confirm a change in concavity.

**Example**
Which of $f(x) = x^4$ and $f(x) = x^5$ has a point of inflexion at $x = 0$?

$f'(x) = 4x^3$
$f''(x) = 12x^2$
$f''(0) = 0$

| $x$ | $0^-$ | $0$ | $0^+$ |
|---|---|---|---|
| $f''(x)$ | + | 0 | + |
| Concavity | up | | up |

No change in concavity

$f'(x) = 5x^4$
$f''(x) = 20x^3$
$f''(0) = 0$

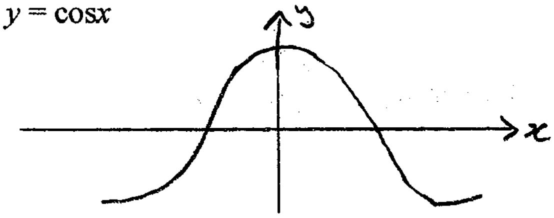
| $x$ | $0^-$ | $0$ | $0^+$ |
|---|---|---|---|
| $f''(x)$ | - | 0 | + |
| Concavity | Down | | up |

Change in concavity
$\Rightarrow$ PoI

*Learning to identify odd and even functions*

- Use substitution of $-x$ to classify functions as odd, even or neither
- Know the symmetry properties of odd and even functions

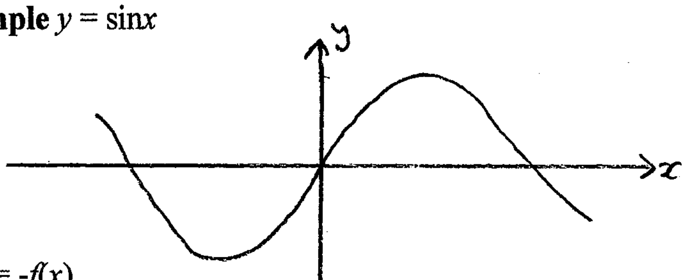
A function is said to **even** if it is symmetrical about the $y$-axis.
**Example** $y = \cos x$




$f(-x) = f(x)$ $\quad f(-x) = \cos(-x) = \cos x = f(x)$

A function is said to be **odd** if it has half-turn symmetry about the origin.

**Example** $y = \sin x$




$f(-x) = -f(x)$

$f(-x) = \sin(-x) = -\sin x = -f(x)$

**Example**

(a) Prove that $f(x) = x^3 - 2x$ is an odd function.
(b) Sketch a graph of $y = f(x)$ for $-2 \le x \le 2$ showing all critical points and the intercepts with the axes.

(a) $f(-x) = (-x)^3 - 2(-x)$
$= -x^3 + 2x$
$= -(x^3 - 2x)$
$= -f(x)$ The function is odd.

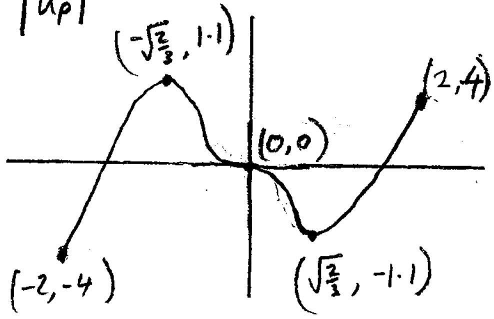
$f'(x) = 3x^2 - 2$ $\quad$ S.P. $f'(x) = 0$ $\quad x = \pm\sqrt{\frac{2}{3}}$

$f''(x) = 6x$ $\quad f''(\sqrt{\tfrac{2}{3}}) > 0 \Rightarrow$ min T.P. $\quad f''(-\sqrt{\tfrac{2}{3}}) < 0 \Rightarrow$ max T.P.

$f''(0) = 0$

| $x$ | $0^-$ | $0$ | $0^+$ |
|---|---|---|---|
| $f''(x)$ | $-$ | | $+$ |
| Concavity | Down | | Up |

Change in concavity $\Rightarrow (0,0)$ is PoI



*Learning to find asymptote*

- Write down vertical asymptotes by identifying values of $x$ for which the function is undefined
- Use polynomial division to find non-vertical asymptotes
- Investigate the behaviour of the function as it approaches an asymptote

**Example**

Find vertical and non-vertical asymptotes to the curve $y = \frac{x^2+1}{x^2}, x \neq 0$.

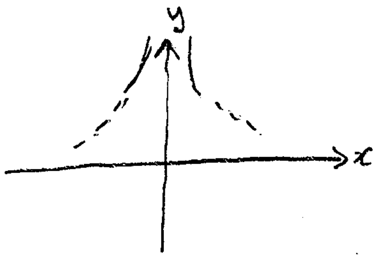
Vertical asymptote at $x = 0$

$y = \dfrac{x^2+1}{x^2} = 1 + \dfrac{1}{x^2}$

$x \to \infty$ $\quad \dfrac{1}{x^2} \to 0$ and $y \to 1$ hence $y = 1$ is a non-vertical asymptote.

$x \to 0$ from left $\quad y \to +\infty$

$x \to 0$ from right $\quad y \to +\infty$

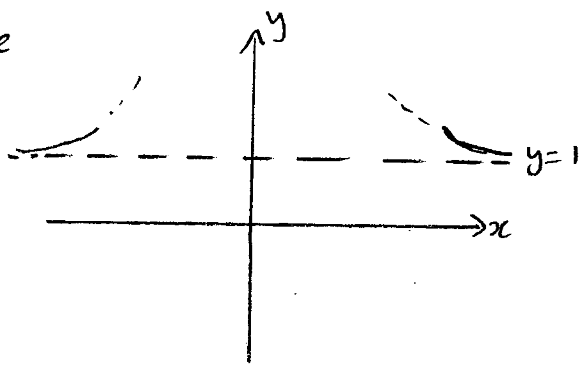



$x \to -\infty \quad y \to 1 + 0^+$

$y \to 1$ from above

$x \to +\infty \quad y \to 1 + 0^+$

$y \to 1$ from above



*Learning to sketch a rational function*

- Find asymptotes
- Find stationary points and determine their nature
- Investigate concavity and identify points of inflexion
- Find intercepts with the axes
- Produce a fully annotated sketch

**Example**

Sketch $y = \frac{x^2+x+2}{x-1}, x \neq 1$.

$x \to 1$ from left $\quad y = \frac{(x+\frac{1}{2})^2 + \frac{7}{4}}{x-1} \to -\infty$

$x \to$ from right $\quad y \to +\infty$

Vertical asymptote $x = 1$

$$\begin{array}{r} x + 2 \\ x-1 \overline{)\, x^2 + x + 2} \\ x^2 - x \\ \hline 2x + 2 \\ 2x - 2 \\ \hline 4 \end{array}$$

$y = x + 2 + \frac{4}{x-1}$ non-vertical asymptote $y = x+2$

$x \to \infty^+ \quad \frac{4}{x-1} \to 0^+ \quad y \to x+2$ from above

$x \to \infty^- \quad \frac{4}{x-1} \to 0^- \quad y \to x+2$ from below

$y = x + 2 + 4(x-1)^{-1}$

$\frac{dy}{dx} = 1 - 4(x-1)^{-2} = 1 - \frac{4}{(x-1)^2}$

S. Ps $\quad 1 - \frac{4}{(x-1)^2} = 0$

$1 = \frac{4}{(x-1)^2}$

$(x-1)^2 = 4$

$x - 1 = \pm 2$

$x = 3$ or $x = -1$

$y = 7 \quad y = -1$

$\frac{d^2y}{dx^2} = 8(x-1)^{-3} = \frac{8}{(x-1)^3}$

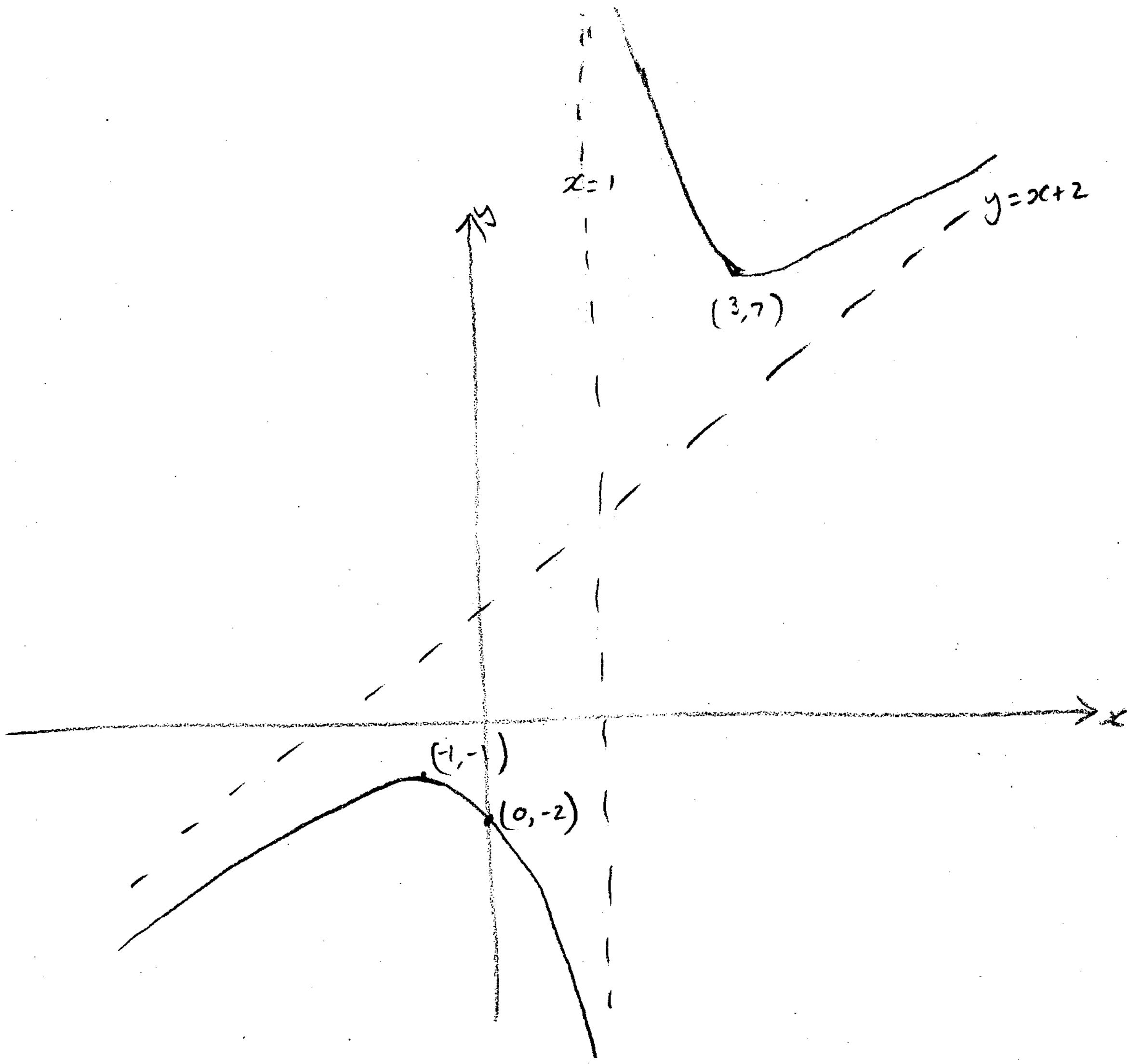
concave up $\forall x > 1$

$x = 3 \quad \frac{d^2y}{dx^2} = \frac{8}{2^3} > 0$ concave up i.e. $x=3$ is min T. P.

$x = -1 \quad \frac{d^2y}{dx^2} = \frac{8}{(-2)^3} < 0$ concave down i.e. $x = -1$ is max T. P. concave down $\forall x < 1$

$x = 0 \quad y = \frac{2}{-1} = -2.$

$y = \frac{(x+\frac{1}{2})^2 + \frac{7}{4}}{x-1} > 0 \quad y \neq 0.$



$y$
$x=1$
$y=x+2$
$(3,7)$
$x$
$(-1,-1)$
$(0,-2)$