1.2 Applying Algebraic skills to sequences and series

Learning the definition of an arithmetic sequence

- Identify an arithmetic sequence
- Use the n^{th} term formula to find a term in the sequence
- Use the formula to find the sum to n terms

An Arithmetic sequence is one with a constant difference between terms
e.g. 1, 4, 7, 10, 13, 16, … $U_n =$

In general, if the first term is a and the constant difference is d $\boxed{U_n = a + (n-1)d}$

Example
Identify the arithmetic sequence with $U_3 = 12$ and $U_{10} = 47$.

$U_3 = a + 2d = 12$

$U_{10} = a + 9d = 47$

$7d = 35$

$d = 5 \quad a = 2$

Sequence is 2, 7, 12, 17, …

Sum to n terms

$S_n = a + a + d + a + 2d + a + 3d + \dots + a + (n-1)d$

Also:

$S_n = a + (n-1)d + a + (n-2)d + a + (n-3)d + \dots + a$

Adding:

$2S_n = 2a + (n-1)d + 2a + (n-1)d + 2a + (n-1)d + \dots + 2a + (n-1)d$

$= n(2a + (n-1)d)$

$$\boxed{S_n = \frac{n}{2}(2a + (n-1)d)}$$

Example
Find the sum of the first 20 terms of the arithmetic sequence with third term = 7, and 10^{th} term = 28.

$U_3 = a + 2d = 7$

$U_{10} = a + 9d = 28$

$7d = 21$

$d = 3 \quad a = 1$

$S_{20} = \frac{20}{2}(2 \times 1 + 19 \times 3) = 590$

Learning the definition of a geometric sequence

- Identify a geometric sequence
- Use the n^{th} term formula to find a term in the sequence
- Use the formula to find the sum to n terms

A geometric sequence is one where the ration of successive terms is constant.
e.g. 1, 3, 9, 27, 81, ... $U_n=$

In general, if the first term is *a* and the common ration is *r* then $\boxed{U_n = a\, r^{n-1}}$

Example

A geometric sequence has first term = 81 and common ration = $\frac{-1}{3}$. Find the 7^{th} term.

$$U_7 = 81 \times \left(\frac{-1}{3}\right)^6 = \frac{81}{3^6} = \frac{1}{9}$$

Sum to n terms

$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$

$rS_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n$

Subtracting: $S_n - rS_n = a - ar^n \qquad S_n(1-r) = a(1-r^n)$

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}}$$

Example

Return to the geometric sequence first term = 81 and common ration = $\frac{-1}{3}$, find the sum of the first (i) 10, (ii) 20, (iii) 100 terms.

$$S_{10} = \frac{81\left(1-\left(\frac{-1}{3}\right)^{10}\right)}{1+\frac{1}{3}} \simeq 60.75$$

$$S_{20} \simeq 60.75$$

$S_{100} = 60.75$ (to 10 sig. figs.)

Sum to infinity

$S_n = \frac{a(1-r^n)}{1-r}$ If $|r| > 1, n \to \infty, r^n \to \infty$ The sum diverges.

If $|r| < 1, n \to \infty, r^n \to 0$ $S_n \to \frac{a}{1-r}$

$$\boxed{S_\infty = \frac{a}{1-r}}$$

Infinite Power Series

- Understand what is meant by an infinite power series
- Understand that a function can be evaluated to <u>any</u> given accuracy using a power series

$1 + x + x^2 + x^3 + x^4 + \cdots$ is a geometric series $a = 1$ and $r = x$

If $|x| < 1$ then S_∞ exists and $S_\infty = \frac{1}{1-x}$

Thus $\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

Learning to use the Maclaurin Expansion

- Understand that, within certain values of x, any function can be expressed as a power series
- Find the Maclaurin expansion for simple functions and their composites
- Understand that there is range of validity and find it for some simple cases

Example

Let $f(x) = e^x = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$

We know $f(0) = e^0 = 1$ so $a_0 = 1$

Differentiating $f'(x) = e^x = a_1 + 2a_2x + 3a_3x^2 + \cdots$

$f'(0) = 1$ so $a_1 = 1$

$f''(x) = e^x = 2a_2 + 2\times 3a_3x + 3\times 4a_4x^2 + \cdots$

$f''(0) = 1$ so $2a_2 = 1$ and $a_2 = \frac{1}{2}$

$f'''(x) = e^x = 2\times 3a_3 + 2\times 3\times 4a_4x + \cdots$

$f'''(0) = 1$ $2\times 3a_3 = 1$ and $a_3 = \frac{1}{2\times 3}$

Continuing we find $a_4 = \frac{1}{2\times 3\times 4}$ $a_5 = \frac{1}{5!}$ etc.

$$\boxed{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots}$$

This is the Maclaurin expansion for e^x.

For what values of x is it valid?

D'Alembert's Ratio Test says that if $\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| < 1$ the series converges.

For the Maclaurin expansion of $e^x \left|\frac{U_{n+1}}{U_n}\right| = \frac{x^{n+1}/(n+1)!}{x^n/n!} =$

For any given x $\lim_{n\to\infty}\frac{x}{n+1} = 0 < 1$so the series converges **for all** values of x.

Exercise
Use the same technique to find expansions for $f(x) = \sin x$ and $f(x) = \cos x$.

Let $f(x) = \sin x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$f(0) = 0 \qquad a_0 = 0$

$f'(x) = \cos x = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$

$f'(0) = 1 \qquad a_1 = 1$

$f''(x) = -\sin x = 2a_2 + 2\times 3 a_3 x + 3\times 4 a_4 x^2 + \cdots$

$f''(0) = 0 \qquad a_2 = 0$

$f'''(x) = -\cos x = 2\times 3a_3 + 2\times 3\times 4 a_4 x + \cdots$

$f'''(0) = -1 \qquad 2\times 3a_3 = -1$

$a_3 = -\frac{1}{3!}$

$f^{iv}(x) = \sin x = 2\times 3\times 4a_4 + 2\times 3\times 4\times 5 a_5 x + \cdots$

$f^{iv}(0) = 0 \qquad a_4 = 0$

$f^{v}(x) = \cos x = 5!\, a_5 + \cdots$

$f^{v}(0) = 1 \qquad a_5 = \frac{1}{5!}$

$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$

Let $f(x) = \cos x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$f(0) = 1 \qquad a_0 = 1$

$f'(x) = -\sin x = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

$f'(0) = 0 \qquad a_1 = 0$

$f''(x) = -\cos x = 2a_2 + 2\times 3 a_3 x + \cdots$

$f''(0) = -1 \qquad 2a_2 = -1$

$a_2 = \frac{-1}{2}$

$f'''(x) = \sin x = 2\times 3\, a_3 + 2\times 3\times 4\, a_4 x + \cdots$

$f'''(0) = 0 \qquad a_3 = 0$

$f^{iv}(x) = \cos x = 4!\, a_4 + \cdots$

$f^{iv}(0) = 1 \qquad a_4 = \frac{1}{4!}$

$\cos x = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \cdots$

General Maclaurin Expansion

Consider the above technique for the general function *f(x)*.

Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$

$f(0) = a_0$

$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$

$f'(0) = a_1$

$f''(x) = 2a_2 + 2\times 3a_3 x + 3\times 4\, a_4 x^2 + \cdots$

$f''(0) = 2a_2 \qquad a_2 = \dfrac{f''(0)}{2}$

$f'''(x) = 2\times 3 a_3 + 2\times 3\times 4\, x + \cdots$

$f'''(0) = 2\times 3 a_3 \qquad a_3 = \dfrac{f'''(0)}{3!}$

$a_4 = \dfrac{f^{iv}(0)}{4!}$

$$\boxed{f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{IV}(0)}{4!}x^4 \cdots}$$

Example

Find the first four terms of the Maclaurin expansion for $(1 + x)^{\frac{1}{3}}$.

$f(x) = (1+x)^{1/3} \qquad f(0) = 1^{1/3} = 1$

$f'(x) = \frac{1}{3}(1+x)^{-2/3} \qquad f'(0) = \frac{1}{3}\cdot 1^{-2/3} = \frac{1}{3}$

$f''(x) = -\frac{2}{9}(1+x)^{-5/3} \qquad f''(0) = -\frac{2}{9}$

$f'''(x) = \frac{10}{27}(1+x)^{-8/3} \qquad f'''(0) = \frac{10}{27}$

$$f(x) = 1 + \frac{1}{3}x - \frac{\overset{1}{\cancel{2}}}{9}\cdot\frac{x^2}{\cancel{2!}} + \frac{\overset{5}{\cancel{10}}}{27}\cdot\frac{x^3}{\underset{3}{\cancel{3!}}} + \cdots$$

$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 + \cdots$$

Composite Functions

The Maclaurin expansion for sin x is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Find a power series for sin ($2x$).

$$\sin 2x = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots$$

$$= 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \ldots$$

$$= 2x - \frac{4x^3}{3} + \frac{4x^5}{15} - \frac{8x^7}{315} + \ldots$$

It is worth learning that:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 \ldots$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots$$