1.3 Applying Algebraic skills to summation and proof

Learning to use sigma notation

- Expand a series given in sigma notation
- Use sigma notation to express a series
- Express $\sum_{r=1}^{n}(2a+b)$ in terms of $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} 1$
- Express $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} 1$ in terms of *n*.

Σ tells us to add up a series of terms. The terms are generated by replacing a variable, usually *r* with natural numbers from a starting point – indicated below the operator – to an end point – indicated above the operator. If the series is infinite, the end point is *n*.

Examples

$$\sum_{1}^{10} 7 = 7 + 7 + \cdots + 7 = 10 \times 7 = 70$$

$$\sum_{1}^{5} 2r^2 = 2 + 8 + 18 + 32 + 50 = 110$$

NB we have used sigma notation for the binomial theorem

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r = {}^nC_o\, x^n y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \cdots + {}^nC_n x^0 y^n$$

$\sum_{r=1}^{n}(ar+b)$ can be expanded and simplified.

$$\sum_{r=1}^{n}(ar+b) = a+b + 2a+b + 3a+b + \cdots + na+b$$

$$= a + 2a + 3a + \cdots + na + b + b + b + \cdots + b$$

$$= a(1+2+3+\cdots+n) + b(1+1+1+\cdots+1)$$

$$= a\sum_{r=1}^{n} r + b\sum_{r=1}^{n} 1$$

Note that $\sum_{r=1}^{n} 1 = n$

$\sum_{r=1}^{n} r$ is the simplest arithmetic series so using the formula for the sum to *n* terms:

$$\sum_{r=1}^{n} r = \frac{n}{2}\left(2\times 1 + (n-1)1\right)$$

$$= \frac{n}{2}(2+n-1)$$

$$= \frac{n}{2}(1+n)$$

$$= \frac{1}{2}n(n+1)$$

Expanding Sums

$$\sum_{r=1}^{n}(ar+b) = a\sum_{r=1}^{n} r + b\sum_{r=1}^{n} 1$$

$$= a \cdot \frac{1}{2}n(n+1) + b \cdot n$$

$$= \frac{a}{2}n(n+1) + bn$$

$$=$$

Example

Find a formula for $\sum_{r=1}^{n}(4r+1)$. $= 4\sum_{r=1}^{n} r + n$

$$= \frac{4}{2}n(n+1) + n$$

$$= 2n(n+1) + n$$

$$= n\left(2(n+1)+1\right)$$

$$= n(2n+3)$$

Standard Formulae

$$\boxed{\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)}$$

$$\boxed{\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2}$$

These formulae can be proved using the method of *proof by induction* which we will look at next.

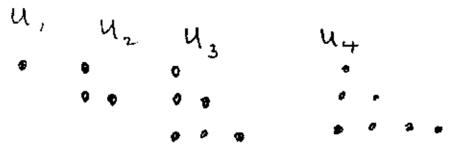
Learning Proof by Induction

- Understand the concept of proof by induction
- Prove n^{th} term formulae by method of proof by induction
- Prove divisibility results
- Prove simple inequalities
- Prove formulae for summations

Example 1

Consider the triangular numbers.

Write a recurrence relation for the sequence and conjecture a formula for the n^{th} term.



$$u_n = u_{n-1} + n$$

$$u_n = \frac{1}{2}n(n+1)$$

$n=1 \quad u_1 = 1 \quad \frac{1}{2} \times 1 \times 2 = 1$ True for $n=1$.

Assume true for $n=k \quad u_k = \frac{1}{2}k(k+1)$

Consider $n=k+1$

$$\begin{aligned} u_{k+1} &= u_k + (k+1) \\ &= \frac{1}{2}k(k+1) + (k+1) \\ &= \frac{1}{2}(k+1)(k+2) \end{aligned}$$

True for $n=k+1$

True for $n=1$ and if true for $n=k$ also true for $n=k+1$.

Hence by induction true $\forall\, n \in \mathbb{N}$.

Example 2
Prove that $2^n > n, \forall n \in \boldsymbol{N}$

$n=1 \quad 2^1 > 1$ True for $n=1$

Assume true for $n=k \quad 2^k > k$

Consider $n=k+1 \quad 2^{k+1} = 2.2^k$

$$2^k > k$$
$$2.2^k > 2k$$

So $2^{k+1} > 2k$

and $2k > k+1$ for $k>1$

Thus $2^{k+1} > k+1$

Inequality is true for $n=1$ and if true $n=k$ then true for $n=k+1$. Hence by induction, true $\forall n \in \mathbb{N}$.

Example 3

Prove that $3^{2n} - 1$ is divisible by 8, $\forall n \in N$

$n=1 \quad 3^2 - 1 = 8$ which is divisible by 8.

Assume true $n=k \quad 3^{2k} - 1 = 8m \quad m \in \mathbb{N}$.

Consider $n=k+1 \quad 3^{2(k+1)} - 1 = 3^{2k+2} - 1$

$$= 3^2 . 3^{2k} - 1$$

But $3^{2k} = 8m + 1 \quad 3^{2(k+1)} - 1 = 3^2(8m+1) - 1$

$$= 3^2 . 8m + 3^2 - 1$$

$$= 3^2 . 8m + 8$$

$$= 8(3^2 m + 1)$$

which is divisible by 8

$3^{2n} - 1$ is divisible by 8 when $n=1$ and if it is divisible by 8 when $n=k$, then it is divisible by 8 when $n=k+1$. Hence by induction, $3^{2n} - 1$ is divisible by 8 $\forall n \in \mathbb{N}$.

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