Sequences and Series 2 - Solutions

1. $f(x) = (x-2)^{-4}$ $\qquad f(0) = \frac{1}{16}$

$f'(x) = -4(x-2)^{-5}$ $\qquad f'(0) = \frac{-4}{-32} = \frac{1}{8}$

$f''(x) = 20(x-2)^{-6}$ $\qquad f''(0) = \frac{20}{64} = \frac{5}{16}$

$f'''(x) = -120(x-2)^{-7}$ $\qquad f'''(0) = \frac{-120}{-128} = \frac{15}{16}$

$f^{iv}(x) = 840(x-2)^{-8}$ $\qquad f^{iv}(0) = \frac{840}{256} = \frac{105}{32}$

$$f(x) = \frac{1}{16} + \frac{1}{8}x + \frac{5}{16}\cdot\frac{x^2}{2!} + \frac{15}{16}\cdot\frac{x^3}{3!} + \frac{105}{32}\cdot\frac{x^4}{4!} + \dots$$

$$= \frac{1}{16} + \frac{1}{8}x + \frac{5}{32}x^2 + \frac{5}{16}x^3 + \frac{35}{256}x^4 + \dots$$

2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

$$\cos(3x) = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \dots$$

$$= 1 - \frac{9x^2}{2} + \frac{27}{8}x^4 - \dots$$

3. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$e^{0.2} = 1 + 0.2 + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{6} + \frac{(0.2)^4}{24} + \dots$$

$$= 1 + 0.2 + \frac{0.04}{2} + \frac{0.008}{6} + \dots$$

$$= 1 + 0.2 + 0.02 + 0.0013\dots$$

$$= 1.221 \text{ (to 3 d.p.)}$$

4. Let $g(x) = \ln(1+x)$ $\qquad g(0) = 0$

$g'(x) = \frac{1}{1+x} = (1+x)^{-1}$ $\qquad g'(0) = 1$

$g''(x) = -(1+x)^{-2}$ $\qquad g''(0) = -1$

$g'''(x) = 2(1+x)^{-3}$ $\qquad g'''(0) = 2$

$g^{iv}(x) = -6(1+x)^{-4}$ $\qquad g^{iv}(0) = -6$

$g^{v}(x) = 24(1+x)^{-5}$ $\qquad g^{v}(0) = 24$

$$g(x) = x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} + \frac{24x^5}{5!} + \cdots$$

$$= x - \frac{1}{2}x^2 + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots$$

$$f(x) = (1+2x)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots\right)$$

$$= x + \frac{3}{2}x^2 - \frac{2}{3}x^3 + \frac{5}{12}x^4 - \frac{3}{10}x^5 + \cdots$$

5. $$\frac{3}{(1+2x)(1-x)} = \frac{A}{1+2x} + \frac{B}{1-x}$$

$$3 = A(1-x) + B(1+2x)$$

Let $x = 1$ $\quad 3 = 3B$ $\quad B = 1$ $\qquad$ Let $x = -\frac{1}{2}$ $\quad 3 = \frac{3}{2}A$ $\quad A = 2$

$$f(x) = \frac{2}{1+2x} + \frac{1}{1-x}$$

$$= 2\left(\frac{1}{1-(-2x)}\right) + \frac{1}{1-x}$$

$$= 2\left(1 + (-2x) + (-2x)^2 + (-2x)^3 + (-2x)^4 + \cdots\right) + 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$= 2\left(1 - 2x + 4x^2 - 8x^3 + 16x^4 + \cdots\right) + 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$= 3 - 3x + 9x^2 - 15x^3 + 33x^4 + \cdots$$