Higher Maths Lechie & Lechie

4A

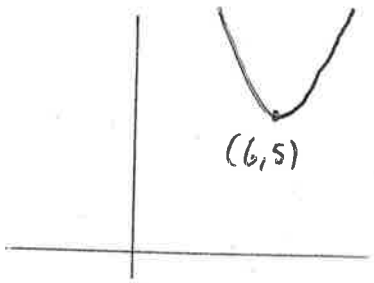
① (a) $y \leq 3$, $y \in \mathbb{R}$

(b) $-1 \leq y \leq 3$, $y \in \mathbb{R}$

(c) $y \geq -2$, $y \in \mathbb{R}$

② (a) $y = (x-6)^2 + 5$

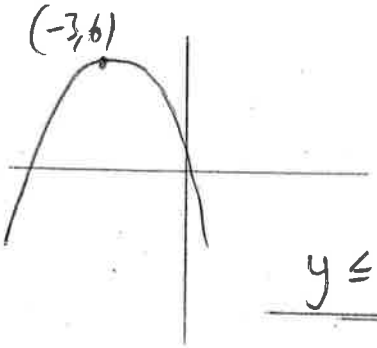
Moves $y = x^2$ → 6 ↑ 5



$y \geq 5$, $y \in \mathbb{R}$

(b) $y = 6 - (x+3)^2$

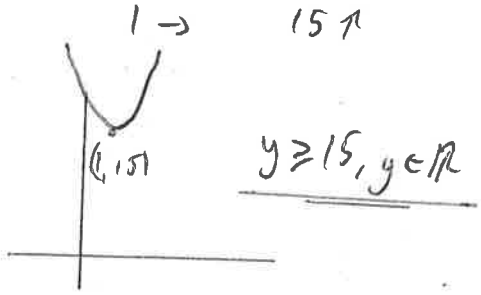
Move $y = x^2$ ← 3 ↑ 6 and turn upside down as $-(x+3)^2$



$y \leq 6$, $y \in \mathbb{R}$

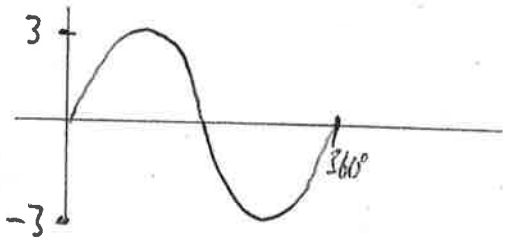
(c) $y = (x-1)^2 + 15$

1 → 15 ↑



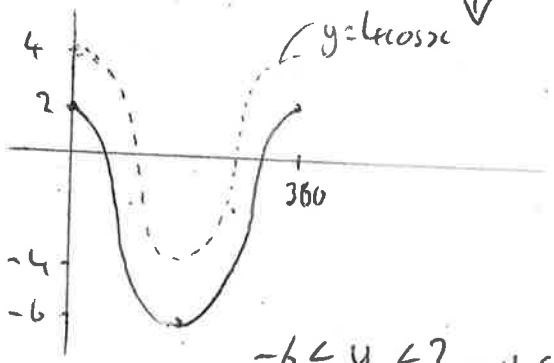
$y \geq 15$, $y \in \mathbb{R}$

(d) $y = 3\sin x$ stretches $y = \sin x$ ↕3



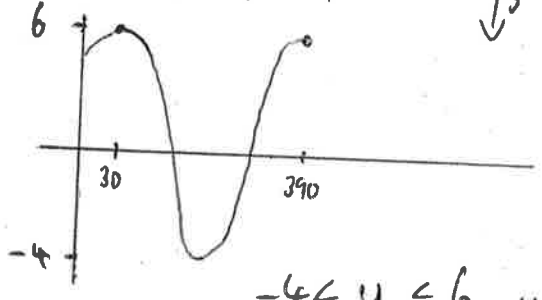
$-3 \leq y \leq 3$, $y \in \mathbb{R}$

(e) $y = 4\cos x - 2$ ↕4 ↓2



$-6 \leq y \leq 2$, $y \in \mathbb{R}$

(f) $y = 5\sin(x-30) + 1$ ↕5 ↑1 →30



$-4 \leq y \leq 6$, $y \in \mathbb{R}$

(3) Domain questions

- Cannot square root a negative
- Cannot divide by zero.

(a) $f(x) = \frac{1}{x-5}$

$x - 5 \neq 0$

$x \neq 5$

(b) $f(x) = \frac{5}{x+9}$

$x + 9 \neq 0$

$x \neq -9$

(c) $f(x) = \sqrt{(x-8)}$

$x - 8 \geq 0$

$x \geq 8$

(d) $f(x) = \sqrt{5-2x}$

$5 - 2x \geq 0$

$5 \geq 2x$

$\frac{5}{2} \geq x$

$x \leq \frac{5}{2}$

(e) $f(x) = \frac{1}{x - 2x^2}$

$x - 2x^2 \neq 0$

$x(x-2) \neq 0$

$x \neq 0, \quad x - 2 \neq 0$

$x \neq 0, \quad x \neq 2$

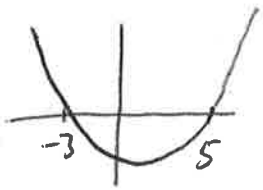
(f) $f(x) = \sqrt{x^2 - 2x - 15}$

$x^2 - 2x - 15 \geq 0$

First factorise to find roots

$(x-5)(x+3) = 0$

roots : $x = 5, \quad x = -3$



sketch shows

$x^2 - 2x - 15 \geq 0$ (i.e. above x-axis)

for $x \leq -3, \quad x \geq 5$

(g) $h(x) = \sqrt{6-x-x^2}$

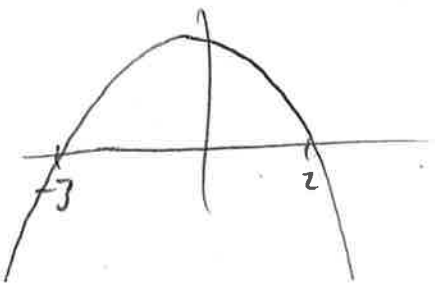
$6-x-x^2 \geq 0$

roots: $6-x-x^2=0$

$0 = x^2+x-6$

$0 = (x+3)(x-2)$

$x=-3,\ x=2$



negative parabola as negative x^2 term.

$6-x-x^2 \geq 0$

$-3 \leq x \leq 2$

(h) $g = \frac{4}{\sqrt{7-x}}$

in this case both not being able to square root a negative and not dividing by zero apply.

$7-x > 0$

$7 > x$

$x < 7$

(i) see (h) for conditions

$h(x) = \frac{1}{\sqrt{2x+3}}$

$2x+3 > 0$

$2x > -3$

$x > -\frac{3}{2}$

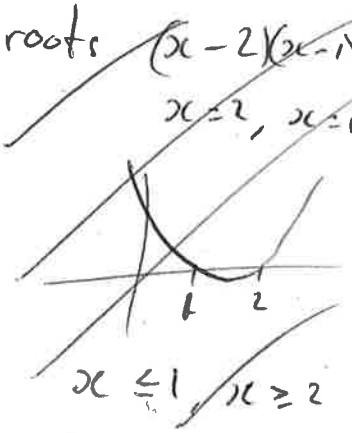
(j) $f(x) = \frac{\sqrt{x}}{x^2-3x+2}$

numerator $\Rightarrow x \geq 0$

denominator $\Rightarrow x^2-3x+2 \neq 0$

roots $(x-2)(x-1)=0$

$x=2,\ x=1$



$x \leq 1,\ x \geq 2$

$(x-2)(x-1) \neq 0$

$x \neq 2,\ x \neq 1$

Hence $x \geq 0,\ x \neq 1,\ x \neq 2$

4B

(a) $f(x) = x+3$, $g(x) = x^2$

$f(g(x))$

$= f(x^2)$

$= x^2+3$

(b) $f(x) = 3x$, $g(x) = x+4$

$f(g(x))$

$= f(x+4)$

$= 3(x+4)$

(c) $f(g(x))$

$= f(3x-2)$

$= 3x-2-1$

$= 3x-3$

(d) $f(g(x))$

$= f(4x)$

$= \sin(4x)$

(e) $f(g(x))$

$= f(2x+3)$

$= 2(x+3)-3$

$= 2x+6-3$

$= 2x+3$

(f) $f(g(x))$

$= f(x^2+5)$

$= 3(x^2+5)-2$

$= 3x^2+15-2$

$= 3x^2+13$

(g) $f(g(x))$

$= (x+4)^2 - 2(x+4)+1$

$= x^2+8x+16-2x-8+1$

$= x^2-6x+9$

(h) $f(g(x))$

$= f(\sin x)$

$= 1-2\sin^2 x$

4B

② (a) $g(f(x))$

$= g(x+3)$

$= (x+3)^2$

(b) $g(f(x))$

$= g(3x)$

$= 3x+4$

(c) $g(f(x))$

$= g(x-1)$

$= 3(x-1)-2$

$= 3x-3-2$

$= 3x-5$

(d) $g(f(x))$

$= g(\sin x)$

$= 4\sin x$

(e) $g(f(x))$

$= g(2x-3)$

$= 2x-3+3$

$= 2x$

(f) $g(f(x))$

$= g(3x-2)$

$= (3x-2)^2+5$

$= (3x-2)(3x-2)+5$

$= 9x^2-6x-6x+4+5$

$= 9x^2-12x+9$

(g) $g(f(x))$

$= g(x^2-2x+1)$

$= 4+x^2-2x+1$

$= x^2-2x+5$

(h) $g(f(x))$

$= g(1-2x^2)$

$= \sin(1-2x^2)$

③ $f(g(x))$

$= f(5x-1)$

$= 3(5x-1)+4$

$= 15x-3+4$

$g(f(x))$

$= g(3x+4)$

(3) continued

$= 5(3x+k) - 1$

$= 15x + 5k - 1$

$f(g(x)) = g(f(x))$

$15x - 3 + k = 15x + 5k - 1$

$-2 + k = 5k$

$-2 = 4k$

$k = -\frac{1}{2}$

$f(g(x)) - g(f(x)) = 0$

$3x^2 + 6x + 5 = 3x^2 + 3$

$6x + 5 = 3$

$6x = 2$

$x = \frac{2}{6}$

$x = \frac{1}{3}$

(4) $f(g(x))$

$= f(x+1)$

$= 3(x+1)^2 + 2$

$= 3(x^2 + 2x + 1) + 2$

$= 3x^2 + 6x + 5$

$g(f(x))$

$= g(3x^2 + 2)$

$= 3x^2 + 2 + 1$

$= 3x^2 + 3$

4B

(5) $f(g(x))$

$h(x) = f(3x-2)$

$h(x) = \frac{1}{3x-2}, \quad x \neq \frac{2}{3}$

$3x - 2 \neq 0$

$3x \neq 2$

$x \neq \frac{2}{3}$

(6) $p(x) = q(r(x))$

$= q(3x+4)$

$= \sqrt{3x+4-3}$

$p(x) = \sqrt{3x+1}, \quad x \geq -\frac{1}{3}$

$3x+1 \geq 0$

$3x \geq -1$

$x \geq -\frac{1}{3}$

(7) $h(x) = f(g(x))$

$= f(3x+1)$

$= \frac{1}{3x+1-5}$

$h(x) = \frac{1}{3x-4}, \quad x \neq \frac{4}{3}$

$3x - 4 \neq 0$

$x \neq \frac{4}{3}$

(8) $f(x) = \frac{x}{2x-1}$

$f(f(x)) = f\left(\frac{x}{2x-1}\right)$

$= \frac{\frac{x}{2x-1}}{2\left(\frac{x}{(2x-1)}\right) - 1}$

$= \frac{\frac{x}{2x-1}}{\frac{2x}{2x-1} - \frac{2x-1}{2x-1}}$

$= \frac{\frac{x}{2x-1}}{\frac{2x-(2x-1)}{2x-1}}$

$= \frac{\frac{x}{2x-1}}{\frac{1}{2x-1}}$

$= \frac{x}{2x-1} \cdot \frac{2x-1}{1}$

$= x$

(9) $h(f(g(x))$

$f(g(x))$

$= f\left(\frac{1}{x}\right)$

$= 1 - \frac{1}{x}$

$h(f(g(x)))$

$= h\left(1 - \frac{1}{x}\right)$

$= \frac{1}{1-\left(1-\frac{1}{x}\right)}$

$= \frac{1}{1 - 1 + \frac{1}{x}}$

$= \frac{1}{\frac{1}{x}}$

$= 1 \times \frac{x}{1}$

$= x$

(10) $g(h(x))$

$= g\left(\frac{1}{x+2}\right)$

$= 3\left(\frac{1}{x+2}\right) + 5$

$= \frac{3}{x+2} + 5$

$= \frac{3}{x+2} + 5\left(\frac{x+2}{x+2}\right)$

$= \frac{3 + 5x + 10}{x+2}$

$= \frac{5x+13}{x+2}$, $x \neq -2$

(11) (a) $h(x) = C(g(x))$

$= f(2x+3)$

$= \frac{1}{(2x+3)^2 - 9}$

$= \frac{1}{(2x+3)(2x+3) - 9}$

$= \frac{1}{4x^2 + 12x + 9 - 9}$

$= \frac{1}{4x^2 + 12x}$

(b) $4x^2 + 12x \neq 0$

$4x(x+3) \neq 0$

$4x \neq 0$, $x+3 \neq 0$

$x \neq 0$, $x \neq -3$

$h(x) = \frac{1}{4x^2+12x}$, $x \neq 0, x \neq -3$

(12) $f(x) = 3x+1$ $g(x) = x^2+1$

$g(f(x))$

$= g(3x+1)$

$= (3x+1)^2 + 1$

$= (3x+1)(3x+1) + 1$

$= 9x^2 + 6x + 1 + 1$

$= 9x^2 + 6x + 2$

$g(f(x)) = 0$

$9x^2 + 6x + 2 = 0$

$b^2 - 4ac$

$= 6^2 - 4(9)(2)$

$= -36$

as $b^2 - 4ac < 0$

$g(f(x)) = 0$ has no real roots.

(12) (b)

$h(x) = x + k$

$h(g(f(x)))$

$= h(9x^2 + 6x + 2)$

$= 9x^2 + 6x + 2 + k$

for equal roots $b^2 - 4ac = 0$

$6^2 - 4(9)(2+k) = 0$

$36 - 36(2+k) = 0$

$36 - 72 - 36k = 0$

$-36 = 36k$

$k = -1$

(13) $V = \pi r^2 h$

$V_t = \pi (0.4t^{1/3})^2 (0.04)$

$V_t = \pi (0.16t^{2/3}) 0.04$

$V_t = 0.064\pi \sqrt[3]{t^2}$ m^3

4B

(14) $C(d(x))$

$= 10(4000 - 200x) + 30\,000$

$= 40000 - 2000x + 30\,000$

$= £(70\,000 - 2000x)$

$r(d(x))$

$= 20$

$= r(4000 - 200x)$

$= 20(4000 - 200x) - 0.005(4000 - 200x)^2$

$= 80\,000 - 4000x - 0.005(16\,000\,000 - 1\,600\,000x + 40\,000x^2)$

$= 80\,000 - 4000x - 80\,000 + 8000x - 200x^2$

$= 200x^2 + 4000x$

Profit = revenue − cost

$= R(d(x)) - C(d(x))$

$= 200x^2 + 4000x - (70\,000 - 2000x)$

Profit $= £(200x^2 + 6000x - 70\,000)$

4C

1(a) $y = 5x - 1$

$5x - 1 = y$

$5x = y + 1$

$x = \frac{1}{5}(y+1)$

$f^{-1}(x) = \frac{1}{5}(x+1)$

(b) $f(x) = 3 - 2x$

$y = 3 - 2x$

$3 - 2x = y$

$3 - y = 2x$

$2x = y - 3$

$x = \frac{1}{2}(y-3)$

$f^{-1}(x) = \frac{1}{2}(x-3)$

(c) $y = \frac{1}{3}x - 1$

$\frac{1}{3}x - 1 = y$

$\frac{1}{3}x = y + 1$

$x = 3(y+1)$

$f^{-1}(x) = 3(x+1)$

(d) $g(x) = 6x - 7$

$6x - 7 = y$

$6x = y + 7$

$x = \frac{1}{6}(y+7)$

$g^{-1}(x) = \frac{1}{6}(x+7)$

(e) $f(x) = 8 - \frac{1}{2}x$

$y = 8 - \frac{1}{2}x$

$\frac{1}{2}x = 8 - y$

$x = 2(8-y)$

$f^{-1}(x) = 2(8-x)$

(f) $y = 7 - 5x$

$5x = y + 7$

$x = \frac{1}{5}(y+7)$

$g^{-1}(x) = \frac{1}{5}(x+7)$

(5) $g(x) = \frac{1}{2}(x+1)$

$\frac{1}{2}(x+1) = y$

$x+1 = 2y$

$x = 2y-1$

$g^{-1}(x) = 2x-1$

(6) $\frac{1}{4}(x+1) = y$

$x+1 = 4y$

$x = 4y-1$

$h^{-1}(x) = 4x-1$

2(a) $7x-1 = y$

$7x = y+1$

$x = \frac{1}{7}(y+1)$

$f^{-1}(x) = \frac{1}{7}(x+1)$

2(b) $f^{-1}(f(x))$

$= f^{-1}(7x-1)$

$= \frac{1}{7}(7x-1+1)$

$= \frac{1}{7}(7x)$

$= x$

$f(f^{-1}(x))$

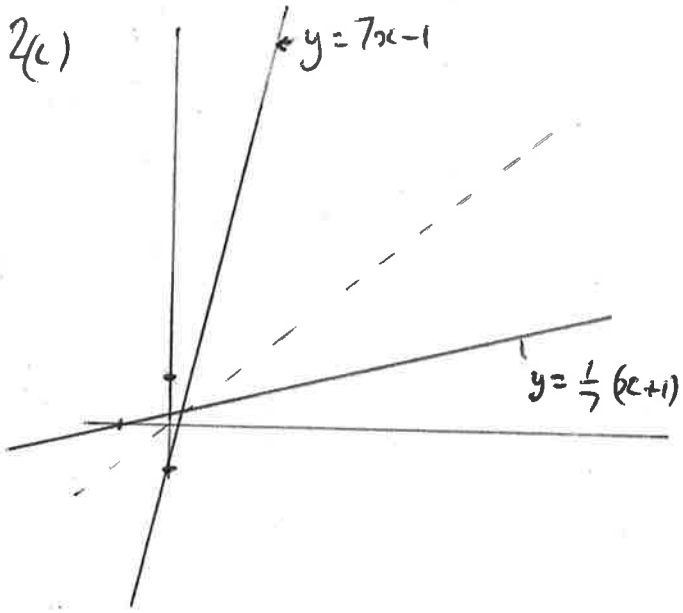
$= f(\frac{1}{7}(x+1))$

$= 7(\frac{1}{7}(x+1)) - 1$

$= x+1-1$

$= x$

2(c)



(3) (a) (i) $f(x) = 3x + 2$

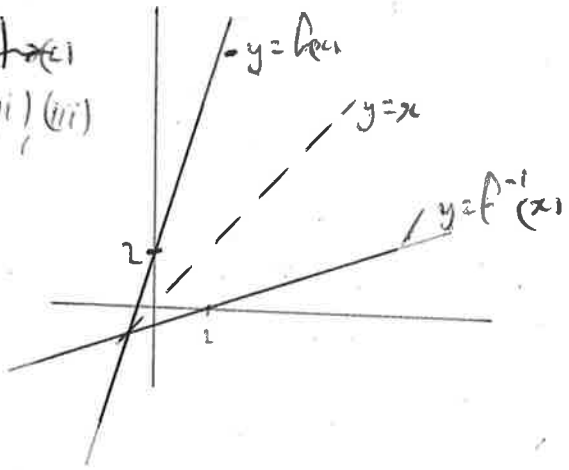
$3x + 2 = y$

$3x = y - 2$

$x = \frac{1}{3}(y - 2)$

$f^{-1}(x) = \frac{1}{3}(x - 2)$

(ii) (iii)



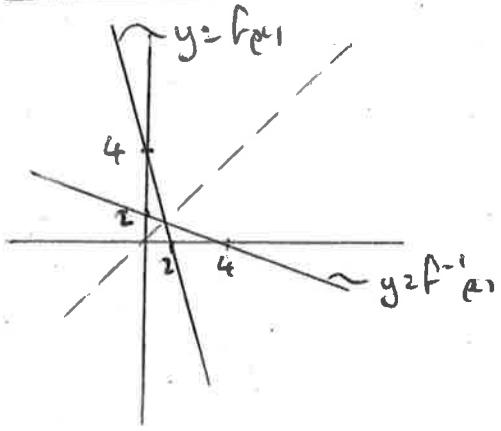
(b) $h(x) = 4 - 2x$

$y = 4 - 2x$

$2x = 4 - y$

$x = \frac{1}{2}(4 - y)$

$h^{-1}(x) = \frac{1}{2}(4 - x)$

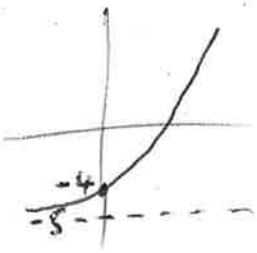


(4) The graph of the inverse of a function is the reflection of the graph of the function in the line $y = x$.

4) (a) $f(x) = 3^x - 5$

moves $y = 3^x$ down by 5

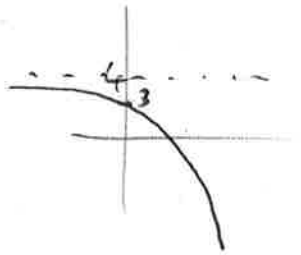
range: $f(x) > -5$



(b) $g = 4 - 5^x$

$y = -5^x$ flips 5^x in x-axis

then move up 4.



$g(x) < 4$

(c) $h = 5^{(x-1)} + 2$

moves 5^x along 1 → and up 2

$h(x) > 2$

(b) $k(x) = \log_7 (x-4)$

4 →

moves $\log_7 x$ 4 to right

Domain $x > 4$

(c) $y = \log_3 (2x-5) + 1$

$= \log_3 (2(x-2.5)) + 1$

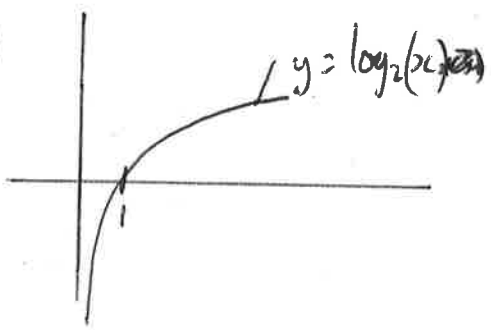
compresses by 2 horizontally

moves 2.5 to right

moves up 1

Domain $x > 2.5$

(2)



$y = \log_2 (x+3)$ moves $\log_2 x$ 3 to the left.

Domain: $x > -3$

(3) $f(x) = 2^x$

$g(x) = x + 4$

$f g(x) = 2^{x+4}$

$= 2^x \cdot 2^4$

$= 2^x \cdot 16$

$= 16(2^x)$

41)

(4) $L(g(x))$

$= f\left(\frac{8}{x^5}\right)$

$= \log_2\left(\frac{8}{x^5}\right)$

$= \log_2 8 - \log_2 x^5$

$= \log_2 2^3 - 5\log_2 x$

$= 3\log_2 2 - 5\log_2 x$

$= \underline{3 - 5\log_2 x}$