Mechanics 1 (AH)

M1.1 Motion in a straight line

M1.1.1 know the meaning of position, displacement, velocity, acceleration, uniform speed, uniform acceleration, scalar quantity, vector quantity

Concepts of position, velocity and acceleration should be introduced using vectors.

Candidates should be very aware of the distinction between scalar and vector quantities, particularly in the case of speed and velocity.

M1.1.2 draw, interpret and use distance/time, velocity/time and acceleration/time graphs

Candidates should be able to draw these graphs from numerical or graphical data.

M1.1.3 know that the area under a velocity/time graph represents the distance travelled

M1.1.4 know the rates of change $v = \frac{dx}{dt} = \dot{x}$ and $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \dot{v} = \ddot{x}$

Candidates should be familiar with the dot notation for differentiation with respect to time.

M1.1.5 derive, by calculus methods, and use the equations governing motion in a straight line with constant acceleration, namely:

$v = u + at$, $s = ut + \frac{1}{2}at^2$ and from these,

$v^2 = u^2 + 2as$, $s = (u + v)t/2$

Candidates need to appreciate that these equations are for motion with *constant* acceleration only. The general technique is to use calculus.

M1.1.6 solve analytically problems involving motion in one dimension under constant acceleration, including vertical motion under constant gravity

M1.1.7 solve problems involving motion in one dimension where the acceleration is dependent on time, ie $a = \frac{dv}{dt} = f(t)$

M1.2 Position, velocity and acceleration vectors including relative motion

M1.2.1 know the meaning of the terms relative position, relative velocity and relative acceleration, air speed, ground speed and nearest approach

M1.2.2 be familiar with the notation:

$\mathbf{r}_P$ for the position vector of P

$\mathbf{v}_P = \dot{\mathbf{r}}_P$ for the velocity vector of P

$\mathbf{a}_P = \dot{\mathbf{v}}_P = \ddot{\mathbf{r}}_P$ for the acceleration vector of P

$\overrightarrow{PQ} = \mathbf{r}_Q - \mathbf{r}_P$ for the position vector of Q relative to P

$\mathbf{v}_Q - \mathbf{v}_P = \dot{\mathbf{r}}_Q - \dot{\mathbf{r}}_P$ for the velocity of Q relative to P

$\mathbf{a}_P - \mathbf{a}_Q = \dot{\mathbf{v}}_P - \dot{\mathbf{v}}_Q = \ddot{\mathbf{r}}_P - \ddot{\mathbf{r}}_Q$ for the acceleration of Q relative to P

M1.2.3 resolve vectors into components in two and three dimensions

This requires emphasis.

M1.2.4 differentiate and integrate vector functions of time

M1.2.5 use position, velocity and acceleration vectors and their components in two and three dimensions; these vectors may be functions of time

M1.2.6 apply position, velocity and acceleration vectors to solve practical problems, including problems on the navigation of ships and aircraft and on the effect of winds and currents

Candidates should be able to solve such problems both by using trigonometric calculations in triangles and by vector components.

Solutions by scale drawing would not be accepted

M1.2.7 solve problems involving collision courses and nearest approach

M1.3 Motion of projectiles in a vertical plane

M1.3.1 know the meaning of the terms projectile, velocity and angle of projection, trajectory, time of flight, range and constant gravity

Candidates also require to know how to resolve velocity into its horizontal and vertical components.

M1.3.2 solve the vector equation $\ddot{\mathbf{r}} = -g\mathbf{j}$ to obtain $\mathbf{r}$ in terms of its horizontal and vertical components

The vector approach is particularly recommended.

M1.3.3 obtain and solve the equations of motion $\ddot{x} = 0$, $\ddot{y} = -g$, obtaining expressions for $\dot{x}$, $\dot{y}$, x and y in any particular case

M1.3.4 find the time of flight, greatest height reached and range of a projectile

Only range on the horizontal plane through the point of projection is required.

M1.3.5 find the maximum range of a projectile on a horizontal plane and the angle of projection to achieve this

M1.3.6 find, and use, the equation of the trajectory of a projectile

Candidates should appreciate that this trajectory is a parabola.

M1.3.9 solve problems in two-dimensional motion involving projectiles under a constant gravitational force and neglecting air resistance

Applications from ballistics and sport may be included and vector approaches should be used where appropriate.

M1.4 Force and Newton's laws of motion

M1.4.1 understand the terms mass, force, weight, momentum, balanced and unbalanced forces, resultant force, equilibrium, resistive forces

M1.4.2 know Newton's first and third laws of motion

M1.4.3 resolve forces in two dimensions to find their components

Resolution of velocities, etc. has been covered in previous sections.

M1.4.4 combine forces to find resultant force

M1.4.5 understand the concept of static and dynamic friction and limiting friction

M1.4.6 understand the terms frictional force, normal reaction, coefficient of friction μ, angle of friction λ, and know the equations $F = \mu R$ and $\mu = \tan\lambda$

Balanced, unbalanced forces and equilibrium could arise here.

Candidates should understand that for stationary bodies, $F \leqslant \mu R$.

M1.4.7 solve problems involving a particle or body in equilibrium under the action of certain forces

Forces could include weight, normal reaction, friction, tension in an inelastic string, etc.

M1.4.8 know Newton's second law of motion, that force is the rate of change of momentum, and derive the equation $\mathbf{F} = m\mathbf{a}$

M1.4.9 use this equation to form equations of motion to model practical problems on motion in a straight line

M1.4.10 solve such equations modelling motion in one dimension, including cases where the acceleration is dependent on time

M1.4.11 solve problems involving friction and problems on inclined planes

Both rough and smooth planes are required.

Mechanics 2 (AH)

M2.1 Motion in a horizontal circle with uniform angular velocity

M2.1.1 know the meaning of the terms angular velocity and angular acceleration

M2.1.2 know that for motion in a circle of radius r, the radial and tangential components of velocity are 0 and $r\dot{\theta}\mathbf{e}_\theta$ respectively, and of acceleration are $-r\dot{\theta}^2\mathbf{e}_r$ and $r\ddot{\theta}\mathbf{e}_\theta$ respectively, where $\mathbf{e}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ and $\mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$ are the unit vectors in the radial and tangential directions, respectively

Vectors should be used to establish these, starting from $\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}$, where r is constant and θ is varying.

M2.1.3 know the particular case where $\theta = \omega t$, ω being constant, when the equations are $\mathbf{r} = r\cos(\omega t)\mathbf{i} + r\sin(\omega t)\mathbf{j}$;
$\mathbf{v} = -r\omega\sin(\omega t)\mathbf{i} + r\omega\cos(\omega t)\mathbf{j}$;
$\mathbf{a} = -r\omega^2\cos(\omega t)\mathbf{i} - r\omega^2\sin(\omega t)\mathbf{j}$;
from which
$v = r\omega = r\dot{\theta}$; $a = r\omega^2 = r\dot{\theta}^2 = v^2/r$ and $\mathbf{a} = -\omega^2\mathbf{r}$

M2.1.4 apply these equations to motion in a horizontal circle with uniform angular velocity including skidding and banking and other applications

Examples should include motion of cars round circular bends, with skidding and banking, the 'wall of death', the conical pendulum, etc.

M2.1.5 know Newton's inverse square law of gravitation, namely that the magnitude of the gravitational force of attraction between two particles is inversely proportional to the square of the distance between the two particles

M2.1.6 apply this to simplified examples of motion of satellites and moons

Circular orbits only.

M2.1.7 find the time for one orbit, height above surface, etc

M2.2 Simple harmonic motion

M2.2.1 know the definition of simple harmonic motion (SHM) and the meaning of the terms oscillation, centre of oscillation, period, amplitude, frequency

M2.2.2 know that SHM can be modelled by the equation $\ddot{x} = -\omega^2 x$

M2.2.3 know the solutions $x = a\sin(\omega t + \alpha)$ and the special cases $x = a\sin(\omega t)$ and $x = a\cos(\omega t)$, of the SHM equation

At this stage these solutions can be verified or established from $\mathbf{r} = a\cos(\omega t)\mathbf{i} + a\sin(\omega t)\mathbf{j}$ rotating round a circle. Solution of second order differential equations is not required.

M2.2.4 know and be able to verify that $v^2 = \omega^2(a^2 - x^2)$, where $v = \dot{x}$; $T = 2\pi/\omega$; maximum speed is ωa, the magnitude of the maximum acceleration is $\omega^2 a$ and when and where these arise

Proof using differential equations is not required here but will arise in the section of work on motion in a straight line later in this unit.

M2.2.5 know the meaning of the term tension in the context of elastic strings and springs

M2.2.6 know Hooke's law, the meaning of the terms natural length, l, modulus of elasticity, λ, and stiffness constant, k, and the connection between them, $\lambda = kl$

M2.2.7 know the equation of motion of an oscillating mass and the meaning of the term position of equilibrium

M2.2.8 apply the above to the solution of problems involving SHM

These will include problems involving elastic strings and springs, and small amplitude oscillations of a simple pendulum but not the compound pendulum.

M2.3 Principles of momentum and impulse

M2.3.1 know that force is the rate of change of momentum

This was introduced in Mechanics 1 (AH).

M2.3.2 know that impulse is change in momentum i.e. $\mathbf{I} = m\mathbf{v} - m\mathbf{u} = \int \mathbf{F}dt$

M2.3.3 understand the concept of conservation of linear momentum

M2.3.4 solve problems on linear motion such as motion in lifts, recoil of a gun, pile-drivers, etc.

The equation $\mathbf{F} = m\mathbf{a}$ is again involved here. Equations of motion with constant acceleration could recur.

M2.4 Principles of work, power and energy

M2.4.1 know the meaning of the terms work, power, potential energy, kinetic energy

M2.4.2 understand the concept of work

Candidates should appreciate that work can be done by or against a force.

M2.4.3 calculate the work done by a constant force in one and two dimensions, ie. $W = Fd$ (one dimension); $W = \mathbf{F}.\mathbf{d}$ (two dimensions)

M2.4.4 calculate the work done in rectilinear motion by a variable force using integration, i.e. $W = \int \mathbf{F}.\mathbf{i}\,dx$; $W = \int \mathbf{F}.\mathbf{v}\,dt$, where $\mathbf{v} = \frac{dx}{dt}\mathbf{i}$

M2.4.5 understand the concept of power as the rate of doing work, i.e. $P = \frac{dW}{dt} = \mathbf{F}.\mathbf{v}$ (constant force), and apply this in practical examples

Examples can be taken from transport, sport, fairgrounds, etc.

M2.4.6 understand the concept of energy and the difference between kinetic (E_K) and potential (E_P) energy

M2.4.7 know that $E_K = \frac{1}{2}mv^2$

M2.4.8 know that the potential energy associated with:

a. a uniform gravitational field is $E_P = mgh$

b. Hooke's law is $E_P = \frac{1}{2}k(\text{extension})^2$

Link with simple harmonic motion.

c. Newton's inverse square law is $E_P = \frac{GMm}{r}$

Link with motion in a horizontal circle.

M2.4.9 understand and apply the work-energy principle

M2.4.10 understand the meaning of conservative forces like gravity, and non-conservative forces like friction

M2.4.11 know and apply the energy equation $E_K + E_P =$ constant, including the situation of motion in a vertical circle

M2.5 Motion in a straight line, where the solution of first order differential equations is required

M2.5.1 know that $a = v\frac{dv}{dx}$ as well as $\frac{dv}{dt}$

M2.5.2 use Newton's law of motion, $\mathbf{F} = m\mathbf{a}$, to form first order differential equations to model practical problems, where the acceleration is dependent on displacement or velocity, i.e. $\frac{dv}{dt} = f(v)$; $v\frac{dv}{dx} = f(x)$; $v\frac{dv}{dx} = f(v)$.

M2.5.3 solve such differential equations by the method of separation of variables

It may be necessary to teach this solution technique, depending on the mathematical background of the candidates.

Examples will be straightforward with integrals which are covered in Mathematics 1, 2 (AH). If more complex, then the anti-derivative will be given.

M2.5.4 derive the equation $v^2 = \omega^2(a^2 - x^2)$ by solving $v\frac{dv}{dx} = -\omega^2 x$

M2.5.5 know the meaning of the terms terminal velocity, escape velocity and resistance per unit mass and solve problems involving differential equations and incorporating any of these terms or making use of $F = \frac{P}{v}$

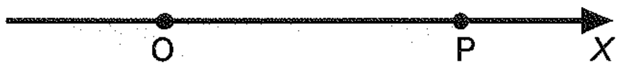
This section can involve knowledge and skills from other

M1.1 MOTION IN A STRAIGHT LINE

Outcome Content
Know the meaning of position, displacement, velocity, uniform speed, uniform acceleration, scalar quantity, vector quantity.
Draw, interpret and use distance/time, velocity/time and acceleration/time graphs.
Know that the area under a velocity/time graph represents the distance travelled.
Know that the rates of change $v = \frac{ds}{dt} = \dot{x}$ and $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{d\dot{x}}{dt} = \dot{v} = \ddot{x}$
Solve problems involving motion in one dimension where the acceleration is dependent on time, i.e. $a = \frac{dv}{dt} = f(t)$.

You should be aware of the distinction between **scalar** and **vector** quantities.
Scalar quantities have magnitude (size) only whereas **vector** quantities have both magnitude and direction.

i.e.	**Scalars**	**Vectors**
	distance	displacement
	speed	velocity
	time	acceleration
	temperature	force



You should know that if a particle P is moving in a straight line along the *X*-axis and $\underline{i}$ is the unit vector in the positive direction of the *X*-axis then:

The **displacement** of P from O at any instant is the vector $\underline{x} = x\underline{i}$, where x = OP.
If P is to the right of O, x is positive and if P is to the left of O, x is negative.

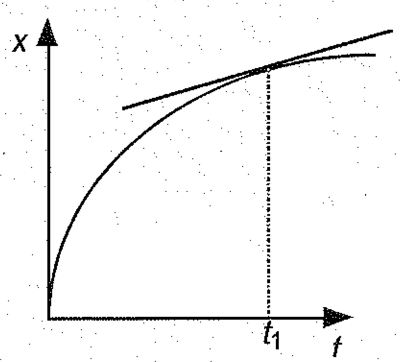
The **velocity** of the particle is the rate of change of its displacement and is the vector $\underline{v} = \frac{dx}{dt}\underline{i}$.

If P is moving to the right $\frac{dx}{dt}$ is positive

and if P is moving to the left $\frac{dx}{dt}$ is negative.

If $|\underline{v}|$, the speed, is denoted by v then $v = \frac{dx}{dt}$ and $x = \int v\,dt$.

$\frac{dx}{dt}$ is sometimes denoted by $\dot{x}$.



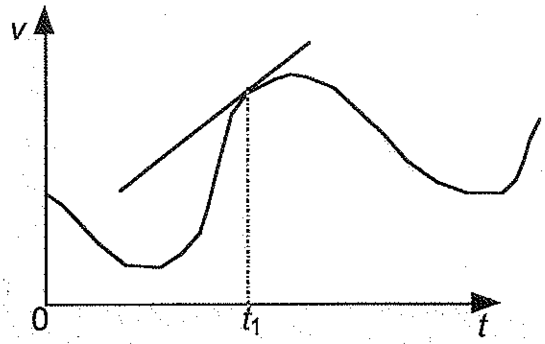
Gradient of graph at $t = t_1$ = $\frac{dx}{dt}$ = velocity at $t = t_1$

The **acceleration** of the particle is the rate of change of its velocity and is the vector $\underline{a} = \frac{dv}{dt}\underline{i}$.

If the velocity is increasing $\frac{dv}{dt}$ is positive and if the velocity is decreasing $\frac{dv}{dt}$ is negative.

If $|\underline{a}|$ is denoted by a then $a = \frac{dv}{dt}$ and $v = \int a\,dt$.

Also $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ and $\frac{d^2x}{dt^2}$ is sometimes denoted by $\ddot{x}$.



Gradient of graph at $t = t_1$ = $\frac{dv}{dt}$ = acceleration at $t = t_1$

Area under graph $t = 0$ to $t = t_1$ = $\int_0^{t_1} v\,dt$ = displacement at $t = t_1$

The area under an acceleration / time graph from t = 0 to $t = t_1$ is

$$\int_0^{t_1} a\,dt = \text{ velocity at } t = t_1.$$

WORKED EXAMPLES

Example 1

A body moves along the X-axis with velocity, measured in ms-1, given by

$$\underline{v} = (3t^2 - 18t + 15)\underline{i}$$

where $\underline{i}$ is the unit vector in the positive direction of the X-axis and t is the time in seconds from the start of the motion.

At the start of the motion the displacement of the body from the origin is 30m. Find:

a) the initial speed of the body;
b) the values of t for which the body is at rest;
c) the acceleration of the body when t = 6;
d) the displacement of the body from O when t =3.

Solution

a) $v = 3t^2 - 18t + 15$

When $t = 0$

$v = 3 \times 0^2 - 18 \times 0 + 15$

$v = 15$ ms^{-1}

b) Body at rest so $v = 0$

$3t^2 - 18t + 15 = 0$

$3(t^2 - 6t + 5) = 0$

$3(t-1)(t-5) = 0$

So at rest when $t = 1$ and 5 s

c) Using $a = \frac{dv}{dt}$ then $a = 6t - 18$

When $t = 6$

$a = 6 \times 6 - 18$

$a = 18$ ms^{-2}

d) Using $v = \frac{dx}{dt}$ and rearranging and integrating gives

$x = \int v\,dt = t^3 - 9t^2 + 15t + c$ where c is the constant of integration

When $t = 0$ $x = 30$ so $c = 30$ and $x = t^3 - 9t^2 + 15t + 30$

When $t = 3$ $x = 3^3 - 9 \times 3^2 + 15 \times 3 + 30 = 21$ m

Displacement from O is 21 m in the positive direction.

Example 2

A body moves along the X-axis from rest at the origin with acceleration, measured in ms^{-2}, given by

$$\underline{a} = (2 - \sqrt{t})\underline{i}$$

where $\underline{i}$ is the unit vector in the positive direction of the X-axis and t is the time in seconds from the start of the motion.

a) Show that the speed increases to a maximum value then decreases. Find this value.

b) Find

i) the time till the body is instantaneously at rest again

ii) the time before it again passes through its starting point.

Solution

a) If a body is accelerating its speed will increase.

For $a = 2 - \sqrt{t}$ $a = 0$ when $2 - \sqrt{t} = 0$

So speed is increasing until $t = 4$.

After $t = 4$ $a < 0$ so body starts to slow down.

There must be a stationary value at $t = 4$ and it is a maximum.

This could be confirmed by using a gradient table.

Using $\frac{dv}{dt} = a$ and rearranging and integrating

$v = \int a dt = \int (2 - \sqrt{t})dt = 2t - \frac{2}{3}t^{\frac{3}{2}} + c$ where c is the constant of integration

When $t = 0$, $v = 0$ so $c = 0$

$v = 2t - \frac{2}{3}t^{\frac{3}{2}}$ and when $t = 4$ $v = 2 \times 4 - \frac{2}{2} \times 4^{\frac{3}{2}} = \frac{8}{3}$ ms^{-1}

b)

i) Body instantaneously at rest so $v = 0$, so $2t - \frac{2}{3}t^{\frac{3}{2}} = 0$ which gives $t = 0$ or $t = 9$ and so it is at rest again after 9 seconds.

ii) At starting point $x = 0$

Using $\frac{dx}{dt} = v$ and rearranging and integrating

$x = \int v dt = \int (2t - \frac{2}{3}t^{\frac{3}{2}})dt = t^2 - \frac{4}{15}t^{\frac{5}{2}} + c$

When $t = 0$, $x = 0$ so $c = 0$

$x = t^2 - \frac{4}{15}t^{\frac{5}{2}}$

When $x = 0$, $t^2 - \frac{4}{15}t^{\frac{5}{2}} = 0$

$$t^2(1 - \frac{4}{15}t^{\frac{1}{2}}) = 0$$

$$t = 0 \text{ or } t^{\frac{1}{2}} = \frac{15}{4}$$

$$t = 0 \text{ or } t = \frac{225}{16} \text{ s}$$

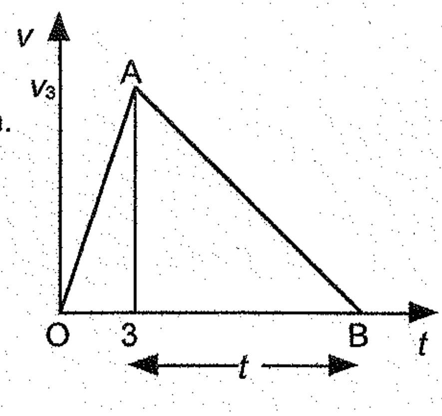
The body passes through the starting point again after about $14 \cdot 06$ seconds.

Example 3
A particle starts from rest at the origin and moves along the X-axis with acceleration, measured in ms^{-2}, given by $\underline{a} = (6 - 2t)\underline{i}$, where $\underline{i}$ is the unit vector in the positive direction of the X-axis and t is the time in seconds from the start of the motion.

Find the maximum speed of the particle.
Find also the displacement of the particle on reaching the maximum speed.
Solution

Using $\frac{dv}{dt} = a$ and rearranging and integrating
$v = \int a\,dt = \int (6-2t)dt = 6t - t^2 + c$
When $t = 0$, $v = 0$ so $c = 0$
$v = 6t - t^2$
Maximum speed occurs when $a = 0$ so $6 - 2t = 0$ gives $t = 3$
Max speed $= 6 \times 3 - 3^2 = 9 \text{ ms}^{-1}$
Using $\frac{dx}{dt} = v$ then $x = \int v\,dt = \int (6t - t^2)dt = 3t^2 - \frac{1}{3}t^3 + c$
Starts from rest so when $t = 0$, $x = 0$ so $c = 0$
$x = 3t^2 - \frac{1}{3}t^3$
At max speed $t = 3$ so $x = 3 \times 3^2 - \frac{1}{3} \times 3^3 = 18$ m

Example 4
A particle starts from rest and moves in a straight line with uniform acceleration $6\,ms^{-2}$ for 3 seconds. It is then brought to rest with uniform acceleration of $-2\,ms^{-2}$.
Draw a velocity/time graph and use it to find the distance travelled by the particle.
Solution
Acceleration is the rate of change of velocity and **uniform** acceleration means **constant** acceleration. This rate of change will be the gradient in the corresponding velocity/time graph.



Gradient OA = acceleration in first 3 seconds
$m_{OA} = \frac{v_3}{3} = 6$ so the maximum speed $v_3 = 18 \text{ ms}^{-1}$
The distance travelled is the area under the velocity / time graph.
So $\frac{v_3}{t} = 2$ where t is the time to come back to rest.
Which gives $\frac{18}{t} = 2$ so $t = 9$ s
Area $\Delta OAB = \frac{1}{2} \times 12 \times 18 = 108$
Distance travelled = 108 m

Exercise M1.1-1

In all of the questions motion is in a straight line, $\underline{i}$ is the unit vector in the positive direction of the X-axis and x, v, a and t have their normal meanings.

1. If $\underline{x} = (t^3 + t)\underline{i}$ find v when $t = 3$ s.
2. If $\underline{x} = (5t^2 - t^3)\underline{i}$ find a when $t = 1$ s.
3. If $\underline{v} = t^3\underline{i}$ find a when $t = 2$ s.
4. If $\underline{v} = (4t + 5)\underline{i}$ and $x = 10$ m when $t = 1$ s, find x when $t = 2$ s.
5. If $\underline{a} = 6t\underline{i}$ and the body is initially at rest, find v when $t = 4$ s.
6. If $\underline{a} = \frac{3}{4}t\underline{i}$ find x when $t = 2$ s given that $v = 4\,\text{ms}^{-1}$, and $x = 10$ m when $t = 4$ s.
7. If $\underline{x} = (t^2 - 3)\underline{i}$ find:
   a) an expression for the velocity of the body at time t
   b) the value of t when the speed is $8\,\text{ms}^{-1}$
   c) the displacement of the body from O when $v = 8\,\text{ms}^{-1}$.
8. If $\underline{x} = (2t^3 - 21t^2 + 60t)\underline{i}$ find:
   a) the values of t when the body is at rest
   b) the initial velocity of the body
   c) an expression for the acceleration of the body at time t
   d) the initial acceleration of the body.
9. If $\underline{v} = (8t - 3t^2)\underline{i}$ and the body is initially at O, find:
   a) an expression for the acceleration of the body at time t
   b) an expression for the displacement of the body from O at time t
   c) how far the body is from O when $t = 3$ s.
10. If $\underline{v} = (3t - 2)(t - 4)\underline{i}$ and $x = 8$ m when $t = 1$ s, find:
    a) the initial velocity of the body
    b) the values of t when the body is at rest
    c) the acceleration of the body when $t = 3$ s
    d) the distance the body is from O when $t = 2$ s.
11. If $\underline{a} = 2t\underline{i}$ and initially the body is at rest at O, find the velocity of the body when $t = 3$ s and the distance the body is then from O.
12. A stone is dropped from a point O at the top of a cliff of height 80 m. The distance x m of the stone below O after t s is given by $x = 5t^2$.
    a) find the velocity of the stone just before it hits the ground
    b) show that the acceleration of the stone is constant.

13. A stone is thrown vertically upwards from a point O. The height x m of the stone above O is given by $x = 20t - 5t^2$.

    a) What is the greatest height above O reached by the stone and how long does it take to reach this height?

    b) What is the initial speed of the stone and what its speed on returning to O?

14. The displacement of a particle from O is given by $\underline{x} = (bt^2 + ct)\underline{i}$, where b and c are constants.

    a) What is the initial displacement of P?

    b) Find an expression for the speed v and determine the initial velocity of P.

    c) Show that a is constant.

    d) By eliminating t from your equations for x and v, show that $v^2 = c^2 + 4bx,\ \ b \neq 0$.

15. A man runs from his house O to the local shop A along a straight road. He realises that he has forgotten to bring any money, so turns round and runs back to O again. During his journey his distance x m from O is given by

$$x = \frac{1}{80}(30t^2 - t^3)$$

    a) Find the distance from O to A.

    b) Find the total time for his journey.

    c) Find his maximum speed during the journey.

16. A body moves along a straight line with acceleration given by $a = \frac{7t}{36}$ where t is the time in seconds. When $t = 0$ the body is at rest at an origin O. The acceleration continues until $t = 6$, when it is then decelerated to rest. During this deceleration $a = -\frac{t}{4}$. Find the value of t when the body comes to rest and the displacement of the body from O at that time.

Answers

1. 28 ms^{-1}

2. 4 ms^{-2}

3. 12 ms^{-2}

4. 21 m

5. 48 ms^{-1}

6. 7 m

7. a) $v = 2t$
b) 4 s
c) 13 m

8. a) 2 and 5 s
b) $60\underline{i}$
c) $a = 12t - 42$
d) $-42\underline{i}$

9. a) $\underline{a} = (8 - 6t)\underline{i}$
b) $\underline{x} = (4t^2 - t^3)\underline{i}$
c) 9 m

10. a) $8\underline{i}$
b) 2/3 and 4 s
c) $4\underline{i}$
d) 2 m

11. a) $\underline{v} = 9\underline{i}$
b) 9 m

12. a) $\underline{v} = 40\underline{i}$
b) $a = 10$ a constant

13. a) 20 m
b) 20 ms^{-1}

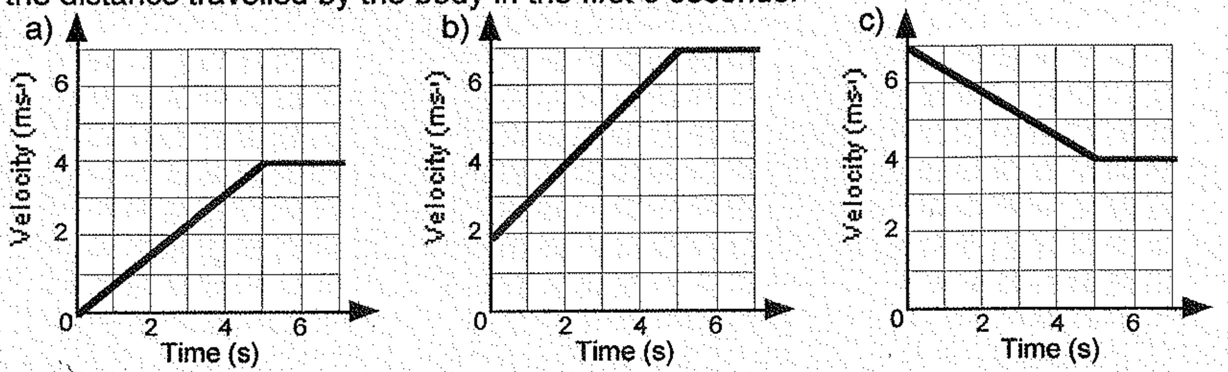
14. a) $\underline{0}$
b) $c\underline{i}$
c) $a = 2b$ which is a constant.

15. a) 50 m
b) 30 s
c) 15/4 ms^{-1}

16. 8 s
$32/3\,\underline{i}$

Exercise M1.1-2

1. Each of the following velocity/time graphs are for a body which accelerates uniformly for a time period of 5 seconds after which it maintains its final velocity. In each case find:
   i) the acceleration of the body during the 5 seconds
   ii) the distance travelled by the body in the first 6 seconds.



2. A cyclist rides along a straight road from point A to point B. He starts from rest, accelerates uniformly to reach a speed of 12 ms^{-1} in 8 seconds. He maintains this speed for a further 20 seconds and then uniformly slows down to rest at B.
   If the whole journey lasts 34 seconds, draw a velocity/time graph and from it find:
   a) his acceleration for the first part of the motion
   b) his acceleration (deceleration) for the last part of the journey
   c) the total distance travelled.

3. A particle is initially at rest at a point A on a straight line ABCD. The particle moves from A to B with uniform acceleration, reaching B with a speed of 12 ms^{-1} after 2 seconds. The acceleration then alters to a constant 1 ms^{-2} and 8 seconds after leaving B the particle reaches C. The particle then slows down uniformly to come to rest at D after a further 10 seconds.
   Draw a velocity/time graph for the motion and from it find:
   a) the acceleration of the particle when travelling from A to B
   b) the speed of the particle on reaching C
   c) the acceleration of the particle when travelling from C to D
   d) the total distance from A to D.

4. Two stations A and B are a distance of $6x$ m apart along a straight track. A train starts from rest at A and accelerates uniformly to a speed v ms^{-1}, covering a distance of x m. The train then maintains this speed until it has travelled a further $3x$ m, it then slows down at a uniform rate to rest at B. Make a sketch of the velocity/time graph for the motion and show that if T is the time taken for the train to travel from A to B then $T = \frac{9x}{v}$ seconds.

5. An elevator travels from rest at the ground floor of a building to rest at the top floor, 50 m above, taking 18 seconds. It accelerates uniformly for 10 m, travels with constant speed for 32 m and then decelerates. Find:
   a) the constant speed
   b) the acceleration and the deceleration.

6. A particle moves in a straight line from rest at A to rest at B. It accelerates uniformly at $a\,\text{ms}^{-2}$, moves with constant speed $V\,\text{ms}^{-1}$ and then decelerates uniformly at $\frac{1}{2}a\,\text{ms}^{-2}$. If the total time for the journey is T s and the distance AB is $\frac{5}{32}aT^2$ m, show that $V = \frac{1}{4}aT\,\text{ms}^{-1}$.

7. A particle moves so that $x = 6t^2 - t^3$.
   a) Show that the particle is initially at O and returns to O after 6 seconds.
   b) Find the maximum displacement of the particle from O in the time interval $0 \le t \le 6$.
   c) Sketch a graph of the speed of the particle against t. Find the maximum speed of the particle in the interval $0 \le t \le 6$.

Answers

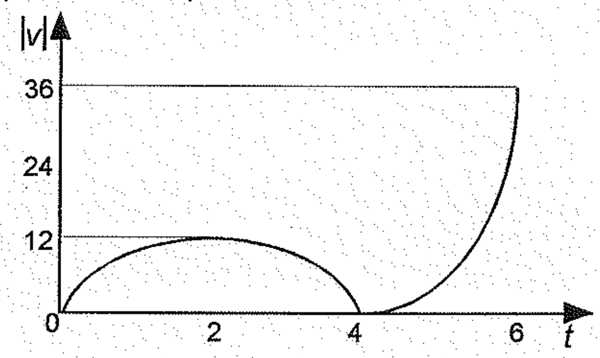
1. a) i) $0{\cdot}8\,\text{ms}^{-2}$ ii) 14 m
b) i) $1\,\text{ms}^{-2}$ ii) 29·5 m
c) i) $-0{\cdot}6\,\text{ms}^{-2}$ ii) 31·5 m

2. a) $1{\cdot}5\,\text{ms}^{-2}$
b) $-2\,\text{ms}^{-2}$
c) 324 m

3. a) $6\,\text{ms}^{-2}$
b) $20\,\text{ms}^{-1}$
c) $-2\,\text{ms}^{-2}$
d) 240 m

5. a) $3{\cdot}\dot{7}\,\text{ms}^{-1}$
b) 0·71 and $0{\cdot}89\,\text{ms}^{-2}$

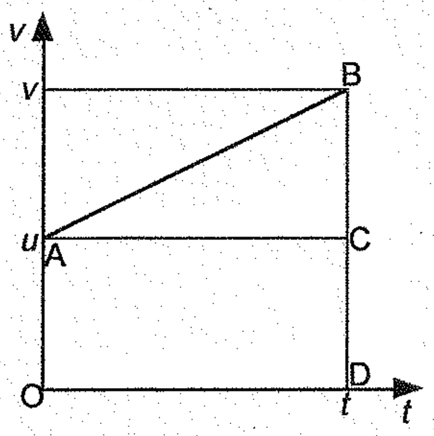
7. b) 32 m
c) Maximum speed is $36\,\text{ms}^{-1}$



Equations of Motion of a body moving in a straight line with line with uniform acceleration - Graphical Approach

The method of obtaining the equations given in this section is for information only, you are required to use calculus methods to derive the equations in this course.

Consider a particle moving in a straight line with constant acceleration so that $\underline{a} = a\underline{i}$ with initial velocity $\underline{u} = u\underline{i}$ and final velocity $\underline{v} = v\underline{i}$ after time t.
The velocity/time graph for such a situation is shown below.



a = rate of change of velocity
= gradient of AB

$= \dfrac{BC}{AC}$

so $a = \dfrac{v-u}{t}$

Rearranging gives $v = u + at$ [1]

Using distance travelled = area under velocity / time graph

$x =$ areaOACD + areaABC

$x = ut + \frac{1}{2}t(v-u)$

but $at = v - u$ from [1]

so $x = ut + \frac{1}{2}at^2$ [2]

It is common in Physics/Mechanics to use s for the distance travelled so the last result is normally seen as

$$s = ut + \tfrac{1}{2}at^2$$

Equations [1] and [2] form the basis for straight line motion uniform acceleration calculations.

Outcome Content

Derive, by calculus methods, and use the equations governing motion in a straight line with constant acceleration, namely:

$v = u + at,\ s = ut + \frac{1}{2}at^2$ and from these

$v^2 = u^2 + 2as,\ s = \dfrac{(u+v)t}{2}$

Solve analytically problems involving motion in one dimension under constant acceleration, including vertical motion under constant gravity.

It has to be clearly emphasised that these equations are **only for motion with constant acceleration**. The general method is to use calculus techniques as covered earlier.

Proof using calculus

Rate of change of velocity = acceleration

so using $\dfrac{dv}{dt} = a$ and rearranging and integrating

$v = \int a\,dt$ but since a is constant

$v = a\int dt$

so $v = at + c$ where c is a constant of integration

When $t = 0$ $v = u$

so $c = u$

which gives $v = u + at$ [1]

Using $\dfrac{dx}{dt} = v$ and rearranging and integrating

$x = \int v\,dt = \int (u + at)\,dt$

so $x = ut + \frac{1}{2}at^2 + c_1$ where c_1 is a constant of integration

When $t = 0$ $x = 0$ so $c_1 = 0$

$x = ut + \frac{1}{2}at^2$ [2]

These are the two equations obtained previously using the graphical approach. Another two equations can be obtained by combining [1] and [2] as follows:

From [1] $t = \dfrac{v-u}{a}$, substituting for t in [2] and replacing x by s

$$s = \frac{u(v-u)}{a} + \frac{1}{2}\frac{(v-u)^2}{a}$$

so $2as = 2u(v-u) + (v-u)^2$

$2as = 2uv - 2u^2 + v^2 - 2uv + u^2$

$2as = v^2 - u^2$

Giving $v^2 = u^2 + 2as$ [3]

Rearranging [3] gives $s = \dfrac{v^2 - u^2}{2a}$ and using the difference of squares

$$s = \frac{(v+u)(v-u)}{2a}$$

From [1] $t = \dfrac{v-u}{a}$ so substituting for $\dfrac{v-u}{a}$ in [3] gives

$s = \dfrac{(u+v)t}{2}$ [4]

WORKED EXAMPLES

Example 1

A particle moving in a straight line with constant acceleration increases its velocity from $4\,ms^{-1}$ to $16\,ms^{-1}$ in 6 seconds. Find the constant acceleration and the distance travelled during the 6 seconds.

Solution

$u = 4\,ms^{-1}$ $\quad v = 16\,ms^{-1}$ $\quad t = 6\,s$

For a use $v = u + at$

$$16 = 4 + 6a$$
$$6a = 12$$
$$a = 2\ ms^{-2}$$

For s use $s = ut + \tfrac{1}{2}at^2$

$$s = 4 \times 6 + \tfrac{1}{2} \times 2 \times 6^2$$
$$s = 60\ m$$

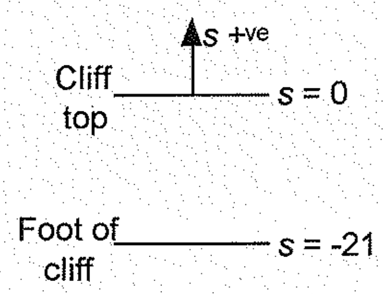
Example 2

A ball is thrown upwards, with a speed $7{\cdot}7\,ms^{-1}$, from the top of a sheer cliff 21 m high in such a way that on the downwards part of the motion the edge of the cliff is just missed and the ball continues downwards. Find:

a) the time taken for the ball to reach the foot of the cliff

b) the velocity of the ball at the instant it hits the ground.

Solution



a) Taking upwards as the positive direction

$u = 7{\cdot}7\ ms^{-1}$ and $a = -9{\cdot}8\ ms^{-2}$

Using $s = ut + \tfrac{1}{2}at^2$

$$-21 = 7{\cdot}7t - 4{\cdot}9t^2$$

so $49t^2 - 77t - 210 = 0$

$$7t^2 - 11t - 30 = 0$$
$$(7t + 10)(t - 3) = 0$$

$t = -\dfrac{10}{7}$ or $t = 3$ but $t > 0$ so $t = 3$

Time to reach foot of cliff is 3 s.

b) Using $v = u + at$

$$v = 7{\cdot}7 - 9{\cdot}8 \times 3$$
$$v = -21{\cdot}7$$

so the velocity is $21{\cdot}7\ ms^{-1}$ vertically downwards.