Madras College Maths Department
Higher Maths
R&C 1.1 Polynomials and Quadratics

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Written solutions for each exercise are available at

http://madrasmaths.com/courses/higher/revison_materials_higher.html

You should check your solutions at the end of each exercise and ask your teacher or attend study support if there any problems.

Polynomials

Polynomials are expressions with one or more terms added together, where each term has a number (called the **coefficient**) followed by a variable (such as *x*) raised to a whole number power. For example:

$$3x^5 + x^3 + 2x^2 - 6 \quad \text{or} \quad 2x^{18} + 10.$$

The **degree** of the polynomial is the value of its highest power, for example:

$3x^5 + x^3 + 2x^2 - 6$ has degree 5 $\qquad$ $2x^{18} + 10$ has degree 18.

Note that quadratics are polynomials of degree two. Also, constants are polynomials of degree zero (e.g. 6 is a polynomial, since $6 = 6x^0$).

Synthetic Division

Synthetic division provides a quick way of evaluating polynomials.

For example, consider $f(x) = 2x^3 - 9x^2 + 2x + 1$. Evaluating directly, we find $f(6) = 121$. We can also evaluate this using synthetic division with detached coefficients.

Step 1

Detach the coefficients, and write them across the top row of the table.

Note that they must be in order of *decreasing* degree. If there is no term of a specific degree, then zero is its coefficient.

	2	–9	2	1

Step 2

Write the number for which you want to evaluate the polynomial (the input number) to the left.

6	2	–9	2	1

Step 3

Bring down the first coefficient.

6	2	–9	2	1
	2			

Step 4

Multiply this by the input number, writing the result underneath the next coefficient.

6	2	–9	2	1
$\times$		12		
	2			

Step 5

Add the numbers in this column.

6	2	–9	2	1
		+ 12		
	2	3		

Repeat Steps 4 and 5 until the last column has been completed.

The number in the lower-right cell is the value of the polynomial for the input value, often referred to as the **remainder**.

6	2	–9	2	1
		12	18	120
	2	3	20	$\boxed{121} = f(6)$

EXAMPLE

1. Given $f(x) = x^3 + x^2 - 22x - 40$, evaluate $f(-2)$ using synthetic division.

Note

In this example, the remainder is zero, so $f(-2) = 0$.

This means $x^3 + x^2 - 22x - 40 = 0$ when $x = -2$, which means that $x = -2$ is a root of the equation. So $x + 2$ must be a factor of the cubic.

We can use this to help with factorisation:

$f(x) = (x+2)(\ q(x)\ )$ where $q(x)$ is a quadratic

Is it possible to find the quadratic $q(x)$ using the table?

Trying the numbers from the bottom row as coefficients, we find:

$$\begin{aligned}&(x+2)(x^2 - x - 20)\\ &= x^3 - x^2 - 20x + 2x^2 - 2x - 40\\ &= x^3 - x^2 - 22x - 40\\ &= f(x).\end{aligned}$$

–2	1	1	–22	–40
		–2	2	40
	1	–1	–20	0

So using the numbers from the bottom row as coefficients has given the correct quadratic. In fact, this method *always* gives the correct quadratic, making synthetic division a useful tool for factorising polynomials.

EXAMPLES

2. Show that $x - 4$ is a factor of $2x^4 - 9x^3 + 5x^2 - 3x - 4$.

3. Given $f(x)=x^3-37x+84$, show that $x=-7$ is a root of $f(x)=0$, and hence fully factorise $f(x)$.

4. Show that $x=-5$ is a root of $2x^3+7x^2-9x+30=0$, and hence fully factorise the cubic.

Using synthetic division to factorise

In the examples above, we have been given a root or factor to help factorise polynomials. However, we can still use synthetic division if we do not know a factor or root.

Provided that the polynomial has an integer root, it will divide the constant term exactly. So by trying synthetic division with all divisors of the constant term, we will eventually find the integer root.

5. Fully factorise $2x^3 + 5x^2 - 28x - 15$.

Note
For ± 1, it is simpler just to evaluate the polynomial directly, to see if these values are roots.

The Factor Theorem and Remainder Theorem

For a polynomial $f(x)$:

If $f(x)$ is divided by $x-h$ then the remainder is $f(h)$, and

$$f(h)=0 \Leftrightarrow x-h \text{ is a factor of } f(x).$$

As we saw, synthetic division helps us to write $f(x)$ in the form

$$(x-h)q(x)+f(h)$$

where $q(x)$ is called the **quotient** and $f(h)$ the **remainder**.

EXAMPLE

7. Find the quotient and remainder when $f(x)=4x^3+x^2-x-1$ is divided by $x+1$, and express $f(x)$ as $(x+1)q(x)+f(h)$.

Finding Unknown Coefficients

Consider a polynomial with some unknown coefficients, such as $x^3 + 2px^2 - px + 4$, where p is a constant.

If we divide the polynomial by $x - h$, then we will obtain an expression for the remainder in terms of the unknown constants. If we already know the value of the remainder, we can solve for the unknown constants.

EXAMPLES

1. Given that $x - 3$ is a factor of $x^3 - x^2 + px + 24$, find the value of p.

Note
This is just the same synthetic division procedure we are used to.

2. When $f(x) = px^3 + qx^2 - 17x + 4q$ is divided by $x - 2$, the remainder is 6, and $x - 1$ is a factor of $f(x)$.

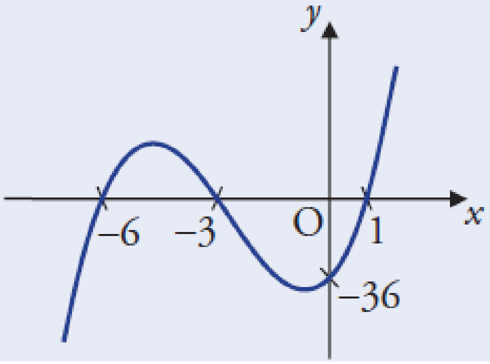
   Find the values of p and q.

Determining the Equation of a Curve

Given the roots, and at least one other point lying on the curve, we can establish its equation using a process similar to that used when finding the equation of a parabola.

EXAMPLE

1. Find the equation of the cubic shown in the diagram below.



Step 1

Write out the roots, then rearrange to get the factors.

Step 2

The equation then has these factors multiplied together with a constant, k.

Step 3

Substitute the coordinates of a known point into this equation to find the value of k.

Step 4

Replace k with this value in the equation.

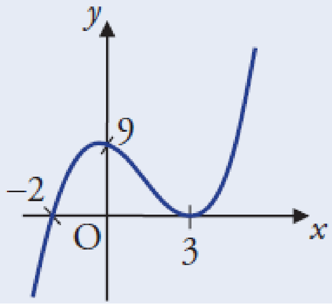
Repeated Roots

If a repeated root exists, then a stationary point lies on the x-axis.

Recall that a repeated root exists when two roots, and hence two factors, are equal.

EXAMPLE

2. Find the equation of the cubic shown in the diagram below.



Note

$x = 3$ is a repeated root, so the factor $(x-3)$ appears twice in the equation.

Using synthetic division to solve equations

We can also use synthetic division to help solve equations.

EXAMPLE

Find the solutions of $2x^3 - 15x^2 + 16x + 12 = 0$.

Finding Intersections of Curves

We have already met intersections of lines and parabolas in this outcome, but we were mainly interested in finding equations of tangents

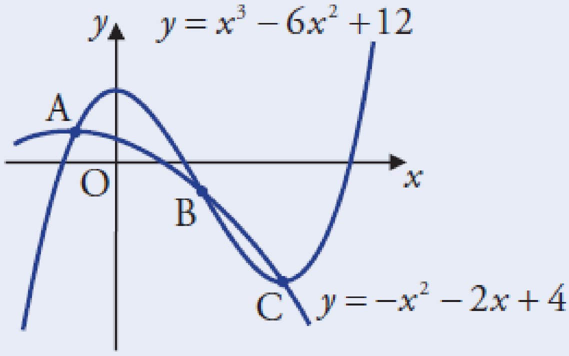
We will now look at how to find the actual points of intersection – and not just for lines and parabolas; the technique works for any polynomials.

EXAMPLES

1. Find the points of intersection of the line $y = 4x - 4$ and the parabola $y = 2x^2 - 2x - 12$.

2. Find the coordinates of the points of intersection of the cubic $y = x^3 - 9x^2 + 20x - 10$ and the line $y = -3x + 5$.

3. The curves $y = -x^2 - 2x + 4$ and $y = x^3 - 6x^2 + 12$ are shown below.



Find the x-coordinates of A, B and C, where the curves intersect.

4. Find the x-coordinates of the points where the curves $y = 2x^3 - 3x^2 - 10$ and $y = 3x^3 - 10x^2 + 7x + 5$.

Solving Quadratic Inequalities

The most efficient way of solving a quadratic inequality is by making a rough sketch of the parabola. To do this we need to know:

- the shape – concave up or concave down,
- the x-axis intercepts.

We can then solve the quadratic inequality by inspection of the sketch.

EXAMPLES

1. Solve $x^2 + x - 12 < 0$.

2. Find the values of x for which $6 + 7x - 3x^2 \geq 0$.

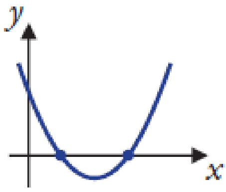
3. Solve $2x^2 - 5x - 3 > 0$.

The Discriminant

Given $ax^2 + bx + c$, we call $b^2 - 4ac$ the **discriminant.**

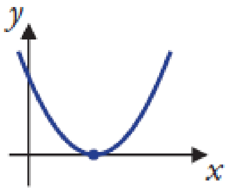
This is the part of the quadratic formula which determines the number of real roots of the equation $ax^2 + bx + c = 0$.

- If $b^2 - 4ac > 0$, the roots are real and unequal (distinct).



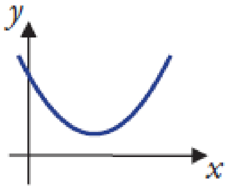
two roots

- If $b^2 - 4ac = 0$, the roots are real and equal (i.e. a repeated root).



one root

- If $b^2 - 4ac < 0$, the roots are not real; the parabola does not cross the x-axis.



no real roots

EXAMPLE

1. Find the nature of the roots of $9x^2 + 24x + 16 = 0$.

2. Find the values of q such that $6x^2 + 12x + q = 0$ has real roots.

3. Find the range of values of k for which the equation $kx^2 + 2x - 7 = 0$ has no real roots.

4. Show that $(2k+4)x^2 + (3k+2)x + (k-2) = 0$ has real roots for all real values of k.

5. Find the values of q for which $x^2 + (q-4)x + \frac{1}{2}q = 0$ has no real roots.

Unit Assessment Practice 1

1 A function f is defined by $f(x) = x^3 + 2x^2 - 5x - 6$ where x is a real number.

(a) (i) Show that $(x+1)$ is a factor of $f(x)$.

(ii) Hence factorise $f(x)$ fully.

[4]

(b) Solve $f(x) = 0$.

[1]

2 The graph of the function $f(x) = kx^2 - 8x + 4$ does not touch or cross the x-axis. What is the range of values for k?

[#2·1, 1]

Unit Assessment Practice 1

1 Solve the equation $x^3 - 4x^2 + x + 6 = 0$

[#2.1+ 5]

2 The function $f(x) = kx^2 - 10x + 12$ has real and equal roots. What is the range of values of k? [2]

Homework 1 – Polynomials and Quadratics

Non Calculator Section

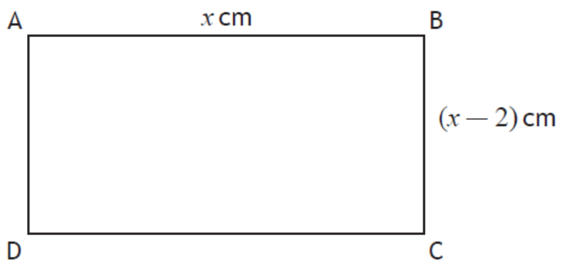
1 Find the range of values for p such that $x^2 - 2x + 3 - p = 0$ has no real roots. **3**

SQA Higher Maths 2016 Paper 2 Question 2

2 Show that $(x + 3)$ is a factor of $x^3 - 3x^2 - 10x + 24$ and hence factorise $x^3 - 3x^2 - 10x + 24$ fully. **4**

SQA Higher Maths 2015 Paper 1 Question 3

3 ABCD is a rectangle with sides of lengths x centimetres and $(x - 2)$ centimetres, as shown.



If the area of ABCD is less than $15\,\text{cm}^2$, determine the range of possible values of x. **4**

SQA Higher Maths 2015 Paper 1 Question 8

4 For the polynomial $6x^3 + 7x^2 + ax + b$,

- $x + 1$ is a factor
- 72 is the remainder when it is divided by $x - 2$.

(*a*) Determine the values of a and b. **4**

(*b*) Hence factorise the polynomial completely. **3**

SQA Higher Maths 2014 Paper 1 Question 22

Additional examples are available in the Dynamic Maths Study Notes available at http://madrasmaths.com/courses/higher/revison_materials_higher.html (password: madrasmaths) and at hsn.uk.net

Unit Assessment 1 Solutions

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & & -1 & -1 & 6 \\ \hline & +1 & +1 & -6 & \boxed{0} \end{array}$$

rem $= 0$ $\therefore$ $x = -1$ roots, $x+1$ factor

$f(x) = (x+1)(x^2+x-6)$

$= (x+1)(x+3)(x-2)$

(b) $f(x) = 0$

$(x+1)(x+3)(x-2) = 0$

$\underline{\underline{x=-1}}$ $\underline{\underline{x=-3}}$ $\underline{\underline{x=2}}$

(2) $b^2-4ac < 0$ for no real roots

$(-8)^2 - 4(k)(4) < 0$

$64 - 16k < 0$

$64 < 16k$

$4 < k$

$\underline{\underline{k > 4}}$

Unit Assessment Practice 2 Solutions

(1)

$$\begin{array}{r|rrrr} -1 & 1 & -4 & 1 & 6 \\ & & -1 & 5 & -6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

rem $= 0$ $x = -1$ root, $x+1$ factor

$(x+1)(x^2-5x+6) = 0$

$(x+1)(x-3)(x-2) = 0$

$\underline{\underline{x=-1}}$ $\underline{\underline{x=3}}$ $\underline{\underline{x=2}}$

(2) $b^2-4ac = 0$ for equal roots

$100 - 48k = 0$

$-48k = -100$

$k = \frac{-100}{-48}$ $\quad$ $\underline{k = \frac{25}{12}}$