Binomial Theorem

Learning to find permutation and combinations

- Calculate permutations and combinations
- Know the definition of ${}^nC_r$
- Recognise $\binom{n}{r}$ notation

Scenario 1

Suppose we arrange the class in a line for a class photograph. How many ways could this be done?

30 choices for the first person
29 choices for the second person
...

Number of ways or **permutations** $= 30 \times 29 \times 28 \times \ldots \times 3 \times 2 \times 1$
$= 30!$

If we wanted just 3 people to represent the class the permutations of 3 from 30 $= 30 \times 29 \times 28$
$= \dfrac{30!}{27!}$

$$\boxed{{}^nP_r = \frac{n!}{(n-r)!}}$$

Examples

1. How many permutations of the letters of MATHS are there?

$${}^5P_5 = 5! = \frac{5!}{0!} = 120$$

$\boxed{\textit{NB}\ 0! = 1 \text{ by definition.}}$

2. How many 4 letter permutations of SCOTLAND?

$${}^8P_4 = \frac{8!}{4!} = \frac{40320}{24} = 1680$$

Scenario 2

Choose 3 colours from red, yellow, green, blue, purple for the new 6th year tie.

The order here is not important.
Suppose the choice is G, B, P. There are 6 ways this could have been arrived at:
GBP, GPB, BGP, BPG, PGB, PBG *i.e.* the number of ways of arranging 3 objects (= 3!)

Number of combinations $= \dfrac{{}^5P_3}{3!} = \dfrac{5!}{3!} = \dfrac{120}{6} = 20$

Algebra: Binomial Theorem and Partial Factions

In general, the number of combinations of r objects from n is $\boxed{{}^nC_r = \dfrac{n!}{(n-r)!\,r!}}$

The notation $\binom{n}{r}$ is common in proof questions so $\boxed{\binom{n}{r} = \dfrac{n!}{(n-r)!\,r!}}$

Learning the binomial theorem

- Know the binomial theorem
- Recognise the connection with Pascal's triangle
- Expand brackets using the binomial theorem

Expand $(x + y)^2$ using the table method.

$\times$	x	y
x	x^2	xy
y	xy	y^2

Multiply answer by $(x + y)$.

	x^2 +	$2xy$ +	y^2
x	x^3	$2x^2y$	xy^2
y	x^2y	$2xy^2$	y^3

And again.

	x^3 +	$3x^2y$ +	$3xy^2$	y^3
x	x^4	$3x^3y$	$3x^2y^2$	xy^3
y	x^3y	$3x^2y^2$	$3xy^3$	y^4

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Coefficients give:

1 1
1 2 1
1 3 3 1
1 4 6 4 1

Algebra: Binomial Theorem and Partial Factions

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Extend Pascal's triangle and write down the expansion of $(x+y)^7$.

$(x+y)^7$

$= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$

Calculate ${}^7C_0 = 1$ ${}^7C_1 = 7$ ${}^7C_2 = 21$ ${}^7C_3 = 35$ ${}^7C_4 = 35$ ${}^7C_5 = 21$ ${}^7C_6 = 7$ ${}^7C_7 = 1$

$$\boxed{(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \ldots + {}^nC_ny^n}$$

This is the **Binomial Theorem**.

The **general term** in the expansion is $\boxed{{}^nC_rx^{n-r}y^r}$

We could use sigma notation to express the expansion

$$\sum_{r=0}^{n} {}^nC_rx^{n-r}y^r$$

Example

$(1+2x)^3$ $= {}^3C_0(1)^3 + {}^3C_1(1)^2(2x)^1 + {}^3C_2(1)^1(2x)^2 + {}^3C_3(2x)^3$

$= 1\times1 + 3\times1\times2x + 3\times1\times4x^2 + 1\times8x^3$

$= 1 + 6x + 12x^2 + 8x^3$

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Algebra: Binomial Theorem and Partial Factions

Learning to apply the binomial theorem

- Write down the general term in an expansion
- Find specific terms in an expansion
- Apply the binomial theorem to harder expansions and to approximations

Examples

1. Write down the general term in the expansion of $\left(2x - \frac{1}{x}\right)^6$ and hence find the term independent of x.

$$^6C_r\ (2x)^{6-r}\left(\frac{-1}{x}\right)^r = {}^6C_r\ 2^{6-r}\ (-1)^r\ x^{6-r}\ x^{-r}$$
$$= {}^6C_r\ 2^{6-r}\ (-1)^r\ x^{6-2r}$$

For independence of x $\quad 6-2r=0 \Rightarrow r=3$

$$^6C_3\ 2^3\ (-1)^3\ x^0 = -160$$

2. Find the x^4 term in the expansion $(1+x)^2(1+2x)^3$

$(1+x)^2 = 1+2x+x^2$

$(1+2x)^3 = 1 + 3(2x) + 3(2x)^2 + (2x)^3 = 1+6x+12x^2+8x^3$

$x^4 = \cancel{1 \times x^4}\quad x \times x^3,\quad x^2 \times x^2,\quad \cancel{x^3 \times x},\quad \cancel{x^4 \times 1}$

$$2x \times 8x^3 + x^2 \times 12x^2 = 28x^4$$

3. Use the binomial theorem to evaluate $(0.98)^7$ to 3 decimal places.

$(0.98)^7 = (1-x)^7$ where $x = 0.02$

$(1-x)^7 = 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 + x^7$

$= 1 - 7(0.02) + 21(0.02)^2 - 35(0.02)^3 + \cdots$

$= 1 - 0.14 + 21 \times 0.0004 - 35 \times 0.000008 + \cdots$

$= 1 - 0.14 + 0.0084 - 0.000280 + \cdots$

$= 0.868$ (to 3 d.p.)

$$\begin{array}{r} 1.00840 \\ -0.14028 \\ \hline 0.86812 \end{array}$$

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p40 Ex 3.6 Q1(a) to (d) and Review Exercise