16A

1(b) $g(x) = x^3 - 3x^2 - 24x + 10$

$g'(x) = 3x^2 - 6x - 24$

$g'(x) = 0$ for stationary points

$3x^2 - 6x - 24 = 0$

$3(x^2 - 2x - 8) = 0$

$3(x+2)(x-4) = 0$

$x + 2 = 0$ $\quad$ $x - 4 = 0$

$x = -2$ $\quad$ $x = 4$

$-1 \le x \le 6$ so only consider $x = 4$

$g(4) = 4^3 - 3(4)^2 - 24(4) + 10$

$= -70$

$(4, -70)$

Nature

x	$\rightarrow$	4	$\rightarrow$
$g'(x) = 3(x+2)(x-4)$	$-$	0	$+$
slope	\	_	/

minimum at $(4, -70)$

16A

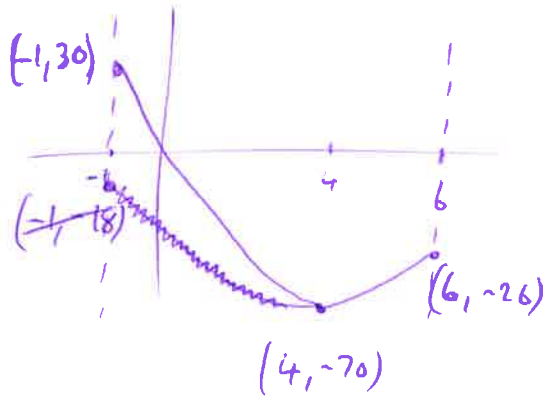
1(b) continued

Find $g(x)$ at end points $(-1 \le x \le 6)$

$$g(-1) = (-1)^3 - 3(-1)^2 - 24(-1) + 10$$
$$= \cancel{-18}\ 30$$

$$g(6) = (6)^3 - 3(6)^2 - 24(6) + 10$$
$$= -26$$

Sketch $g(x)$ $(-1 \le x \le 6)$



Min value of -70 at $x = 4$

Max value of 30 at $x = -1$

16A

2 (a) $f(x) = (x+3)(x-1)^2$

x-int when $f(x) = 0$

$(x+3)(x-1)^2 = 0$

$x = -3 \quad x = 1$

x-ints $(-3, 0)$ $(1, 0)$

y-int when $x = 0$

$f(0) = (0+3)(0-1)^2$

$= 3$

y-int $(0, 3)$

(b) $f(x) = (x+3)(x^2 - 2x + 1)$

$= x^3 - 2x^2 + x + 3x^2 - 6x + 3$

$= x^3 + x^2 - 5x + 3$

$f'(x) = 3x^2 + 2x - 5$

$f'(x) = 0$ for stationary points

$3x^2 + 2x - 5 = 0$

$(3x+5)(x-1) = 0$

$3x + 5 = 0 \qquad x - 1 = 0$

$3x = -5 \qquad x = 1$

$x = -\frac{5}{3}$

16A

2(b) continued

$$f\left(-\frac{5}{3}\right) = \left(-\frac{5}{3}+3\right)\left(-\frac{5}{3}-1\right)^2$$
$$= \frac{256}{27}$$
$$= 9\tfrac{13}{27}$$

$\left(-\frac{5}{3}, 9\tfrac{13}{27}\right)$

$$f(1) = (1+3)(1-1)^2$$
$$= 0$$

$(1, 0)$

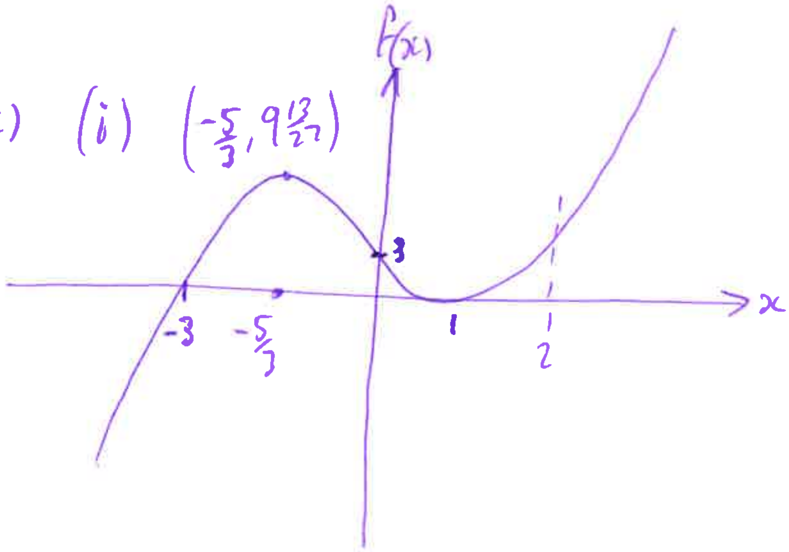
Nature

x	$\to$	$-\frac{5}{3}$	$\to$	1	$\to$
$f'(x) = (3x+5)(x-1)$	$+$	0	$-$	0	$+$
Slope	/	—	\	—	/

max @ $\left(-\frac{5}{3}, 9\tfrac{13}{27}\right)$ min @ $(1, 0)$

16 A

2 (c) (i) $\left(-\frac{5}{3}, 9\frac{13}{27}\right)$



(c) (ii) $f(0) = 3$

$f(2) = (2+3)(2-1)^2$

$= 5$

$0 \le x \le 2$ min value $= 0$ at $x = 1$

max value $= 5$ at $x = 2$

16A

5(c) $2\sqrt{3}\cos\left(x-\frac{\pi}{3}\right)$

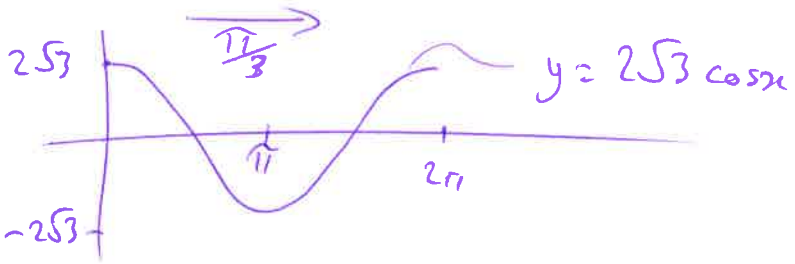
$$= 2\sqrt{3}\left(\cos x\cos\frac{\pi}{3} + \sin x\sin\frac{\pi}{3}\right)$$

$$= 2\sqrt{3}\left(\cos x\cdot\frac{1}{2} + \sin x\cdot\frac{\sqrt{3}}{2}\right)$$

$$= \frac{2\sqrt{3}}{2}\cos x + \frac{2\sqrt{3}\sqrt{3}}{2}\sin x$$

$$= \underline{\underline{3\sin x + \sqrt{3}\cos x}}$$

5(d) $y = 2\sqrt{3}\cos\left(x-\frac{\pi}{3}\right)$



$\xrightarrow{\pi/3}$

max value of $y = 2\sqrt{3}\cos\left(x-\frac{\pi}{3}\right)$ is $2\sqrt{3}$ at $x = \frac{\pi}{3}$

min value of $y = 2\sqrt{3}\cos\left(x-\frac{\pi}{3}\right)$ is $-2\sqrt{3}$ at $x = \frac{4\pi}{3}$

$\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

$f\left(\frac{\pi}{2}\right) = 3\sin\frac{\pi}{2} - \sqrt{3}\cos\frac{\pi}{2}$
$= 3$

$f\left(\frac{3\pi}{2}\right) = 3\sin\frac{3\pi}{2} - \sqrt{3}\cos\frac{3\pi}{2}$
$= -3$

<u>Max value</u> ~~at x = 3π/2~~ of <u>3</u> at $x = \frac{\pi}{2}$

<u>Min value</u> of <u>$-2\sqrt{3}$</u> at $x = \frac{4\pi}{3}$

16A

5(a) (ii)

x	$\rightarrow$	$\frac{\pi}{3}$	$\rightarrow$	$\frac{4\pi}{3}$	$\rightarrow$
$f'(x) = 3\cos x - \sqrt{3}\sin x$	$+$	0	$-$	0	$+$
slope	/	—	\	—	/

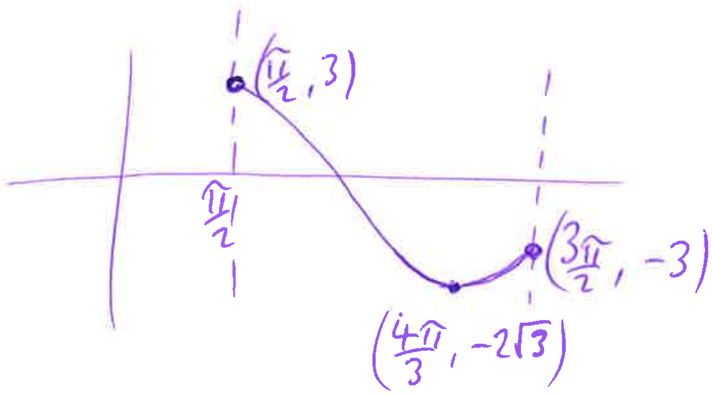
max @ $\left(\frac{\pi}{3}, 2\sqrt{3}\right)$ min @ $\left(\frac{4\pi}{3}, -2\sqrt{3}\right)$

5(b) $f\left(\frac{\pi}{2}\right) = 3\sin\frac{\pi}{2} + \sqrt{3}\cos\frac{\pi}{2}$

$= 3$

$f\left(\frac{3\pi}{2}\right) = 3\sin\frac{3\pi}{2} + \sqrt{3}\cos\frac{3\pi}{2}$

$= -3$



$\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$ greatest value $= 3$ at $x = \frac{\pi}{2}$

least value $= -2\sqrt{3}$ at $x = \frac{4\pi}{3}$

16A

⑤ (a) $f(x) = 3\sin x + \sqrt{3}\cos x$

$f'(x) = 3\cos x - \sqrt{3}\sin x$

$f'(x) = 0$ for stationary points

$3\cos x - \sqrt{3}\sin x = 0$

$3\cos x = \sqrt{3}\sin x$

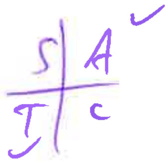
$3 = \sqrt{3}\dfrac{\sin x}{\cos x}$

$\dfrac{3}{\sqrt{3}} = \dfrac{\sin x}{\cos x}$

$\sqrt{3} = \tan x$

(* could use wave function here but as equal to 0 we can rearrange to get $\tan x$).

$x = \dfrac{\pi}{3}, \dfrac{4\pi}{3}$ radians



$f\left(\frac{\pi}{3}\right) = 3\sin\frac{\pi}{3} + \sqrt{3}\cos\frac{\pi}{3}$

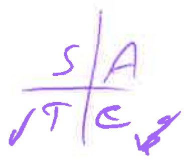
$= 3\frac{\sqrt{3}}{2} + \sqrt{3}\cdot\frac{1}{2}$

$= 2\sqrt{3}$

$\left(\frac{\pi}{3}, 2\sqrt{3}\right)$

$f\left(\frac{4\pi}{3}\right) = 3\sin\left(\frac{4\pi}{3}\right) + \sqrt{3}\cos\left(\frac{4\pi}{3}\right)$

$= -3\sin\pi/3 - \sqrt{3}\cos\pi/3$



$= -3\frac{\sqrt{3}}{2} - \sqrt{3}\cdot\frac{1}{2}$

$= -2\sqrt{3}$

$\left(\frac{4\pi}{3}, -2\sqrt{3}\right)$

16B

(1) $V(x) = x^3 - 105x^2 + 3000x + 2000$

$V'(x) = 3x^2 - 210x + 3000$

$V'(x) = 0$ for stationary points

$3(x^2 - 70x + 1000) = 0$

$3(x-20)(x-50) = 0$

$x = 20 \quad x = 50$

$V(20) = (20)^3 - 105(20)^2 + 3000(20) + 2000$

$= £28,000$

$V(50) = (50)^3 - 105(50)^2 + 3000(50) + 2000$

$= £14,500$

x	$\rightarrow$	20	$\rightarrow$	50	$\rightarrow$
$V'(x) = 3(x-20)(x-50)$	+	0	−	0	+
Slope	/	—	\	—	/

local max value of £28,000 ~~when~~ after 20 days

local min value of £14,500 after 50 days

16B

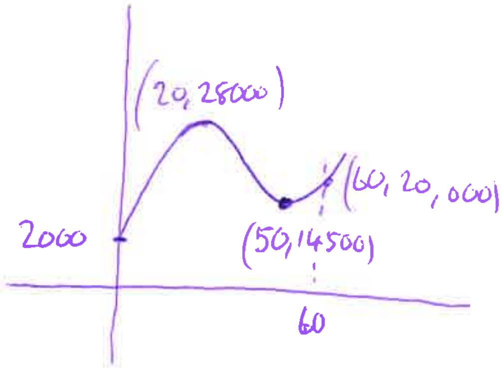
① continued

$0 \le x \le 60$ so consider $V(0)$ & $V(60)$ also.

$V(0) = 2000$

$V(60) = (60)^3 - 105(60)^2 + 3000(60) + 2000$

$= £20,000$

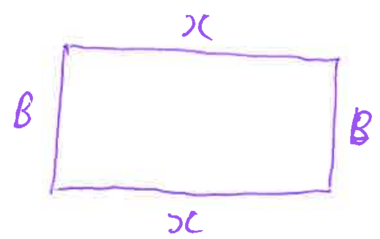


Max value $= £28,000$ after 20 days

Min value $= £2,000$ which was initial value,

16B

② (a)



$P = 60cm$

$$2x + 2B = 60$$
$$2B = 60 - 2x$$
$$B = 30 - x$$

(b) $A(x) = LB$
$$= x(30 - x)$$
$$A(x) = 30x - x^2$$

(c) $A'(x) = 30 - 2x$

$A'(x) = 0$ for stationary points

$$30 - 2x = 0$$
$$30 = 2x$$
$$x = 15cm$$

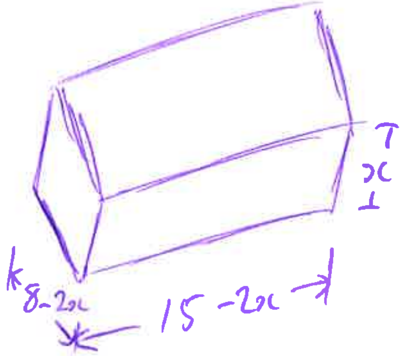
x	$\rightarrow$	15	$\rightarrow$
$A'(x) = 30 - 2x$	+	0	−
Slope	/	—	\

max value when $x = 15cm$

$$A(15) = 30(15) - 15^2$$
$$= 225cm^2$$

16B

3



a length of x was cut from each end of the length & width to create the sides of height x

$$V = LBH$$

$$= (15-2x)(8-2x)x$$

$$= (120 - 30x - 16x + 4x^2)x$$

$$= 4x^3 - 46x^2 + 120x$$

$$V'(x) = 12x^2 - 92x + 120$$

$V'(x) = 0$ for stationary points

$$\cancel{12x^2 - 92x + 120 = 0}$$

$$\cancel{12(x^2 - 5x + 10) = 0}$$

$$12x^2 - 92x + 120 = 0$$

$$4(3x^2 - 23x + 30) = 0$$

$$4(3x - 5)(x - 6) = 0$$

$3x - 5 = 0$ $\qquad$ $x - 6 = 0$

$x = \frac{5}{3}$ $\qquad$ $x = 6$

16B

(3) continued

Width $= 8-2x$ discount $x=6$ as this would give a width of -4 cm,

x	$\rightarrow$	$5/3$	$\rightarrow$
$V'(x) = 4(3x-5)(x-6)$	$+$	0	$-$
Slope	/	—	\

max volume when $\underline{\underline{x = \frac{5}{3} \text{ inches}}}$

$$V\left(\frac{5}{3}\right) = 4\left(\frac{5}{3}\right)^3 - 46\left(\frac{5}{3}\right)^2 + 120\left(\frac{5}{3}\right)$$

$$= \underline{\underline{90.74 \text{ inch}^2}}$$

16B

(4) (a) $V = 3375\text{ ml}$

$= 3375\text{ cm}^3$

$V = x^2h$

$\Rightarrow \underline{x^2h = 3375}$ (1)

$S.A. = x^2 + x^2 + 4xh$

$= \underline{2x^2 + 4xh}$

using (1) $h = \dfrac{3375}{x^2}$

$S.A. = 2x^2 + 4x\left(\dfrac{3375}{x^2}\right)$

$= 2x^2 + \dfrac{13500}{x}$

(b) $A(x) = 2x^2 + 13500x^{-1}$

$A'(x) = 4x - 13500x^{-2}$

$A'(x) = 0$ for stationary points

$4x - \dfrac{13500}{x^2} = 0$

16B

(4)(b) continued

$$4x = \frac{13500}{x^2}$$

$$4x^3 = 13500$$

$$x^3 = 3375$$

$$x = \sqrt[3]{3375}$$

$$x = 15\text{ cm}$$

x	$\rightarrow$	15	$\rightarrow$
$A'(x) = 4x - \frac{13500}{x^2}$	$-$	0	$+$
Slope	$\searrow$	—	$\nearrow$

minimum surface area when $x = 15\text{cm}$

(c) $A(15) = 2(15)^2 + \frac{13500}{(15)}$

$= 1350\text{ cm}^2$

16B

(a) $V = \pi r^2 h$

$216 = \pi r^2 h$ (1)

$1m^3 = (100 \, cm)^3$

$= 1\,000\,000 \, cm^3$

$= 1\,000\,000 \, ml$

$1m^3 = 1000$ litres

$216 m^3 = 216000$ litres

$S \cdot A \cdot =$ top + bottom + curved area

$= 2\pi r^2 + \pi d h$

$= 2\pi r^2 + 2\pi r h$

from (1) $h = \frac{216}{\pi r^2}$

$S \cdot A \cdot = 2\pi r^2 + 2\pi r \left(\frac{216}{\pi r^2}\right)$

$= 2\pi r^2 + \frac{432}{r}$

(b) (i) $A(r) = 2\pi r^2 + 432 r^{-1}$

$A'(r) = 4\pi r - 432 r^{-2}$

$A'(r) = 0$ for stationary points

$4\pi r - \frac{432}{r^2} = 0$

16B

6(b) continued

$$4\pi r - \frac{432}{r^2} = 0$$

$$4\pi r = \frac{432}{r^2}$$

$$4\pi r^3 = 432$$

$$r^3 = \frac{432}{4\pi}$$

$$r = \sqrt[3]{\frac{432}{4\pi}}$$

$$\underline{\underline{r = 3.25\text{m}}}$$

r	$\rightarrow$	3.25	$\rightarrow$
$A'(r) = 4\pi r - \frac{432}{r^2}$	$-$	0	$+$
Slope	\	—	/

<u>minimum</u> surface area when $\underline{\underline{x = 3.25\text{m}}}$

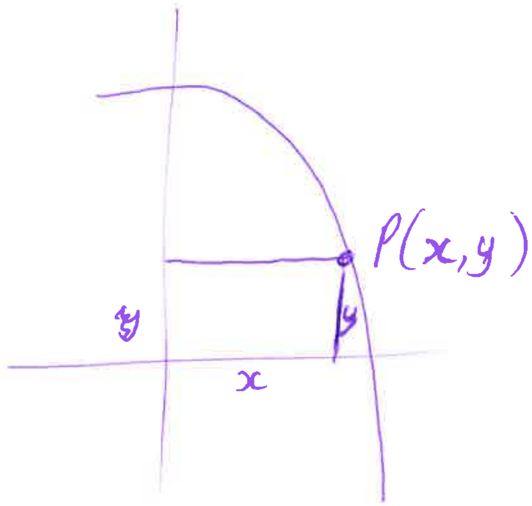
6(b)(ii) $A(3.25) = 2\pi(3.25)^2 + \frac{432}{3.25}$

minimum S.A. $= \underline{\underline{199.3\text{m}^2}}$

16B

9. $A = LB$

$= xy$



we know $y = 8 - x^3$

$A = x(8 - x^3)$

$= 8x - x^4$

$A'(x) = 8 - 4x^3$

$A'(x) = 0$ for stationary point

$8 - 4x^3 = 0$

$8 = 4x^3$

$x^3 = 2$

$x = \sqrt[3]{2}$

x	$\rightarrow$	$\sqrt[3]{2}$	$\rightarrow$
$A'(x) = 8 - 4x^3$	$+$	0	$-$
	/	—	\

max area when $x = \sqrt[3]{2}$

$A(\sqrt[3]{2}) = \sqrt[3]{2}\left(8 - (\sqrt[3]{2})^3\right)$

$= 6\sqrt[3]{2}$

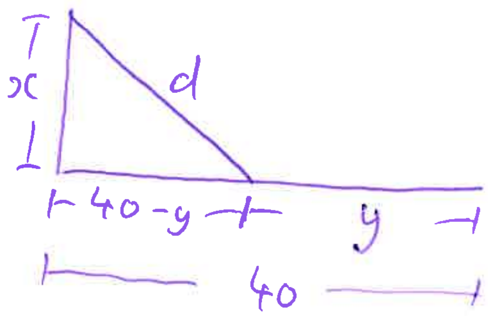
$= 7.56$ square units

16B

12 (a) $S = DT$

$x = 60t \qquad y = 80t$

(b)



$(d(t))^2 = x^2 + (40-y)^2$

$(d(t))^2 = (60t)^2 + (40-80t)^2$

$(d(t))^2 = 3600t^2 + 1600 - 6400t + 6400t^2$

$d(t)^2 = 10\,000t^2 - 6400t + 1600$

$= 400(25t^2 - 16t + 4)$

$d(t) = \sqrt{400}\sqrt{25t^2-16t+4}$

$= 20\sqrt{25t^2-16t+4}$

(c) $d(t) = 20(25t^2-16t+4)^{1/2}$

$d'(t) = 20 \times \frac{1}{2}(25t^2-16t+4)^{-1/2} \times \frac{d}{dt}(25t^2-16t+4)$

$= \dfrac{10(50t-16)}{\sqrt{25t^2-16t+4}}$

12(c) continued

$d'(t) = 0$ for stationary point

$$\frac{10(50t-16)}{\sqrt{25t^2-16t+4}} = 0$$

$$10(50t-16) = 0$$

$$50t - 16 = 0$$

$$50t = 16$$

$$t = \frac{16}{50} \text{ hours} = \underline{\underline{19.2 \text{ minutes}}}$$

$$= \underline{\underline{19 \text{ minutes}}}$$

t	$\rightarrow$	$\frac{16}{50}$	$\rightarrow$
$d'(t) = \frac{10(50t-16)}{\sqrt{25t^2-16t+4}}$	$-$	0	$+$
slope	\	—	/

<u>minimum distance</u> at $t = \frac{16}{50}$ hour <u>(19 mins)</u>

(note: answer in textbook answers is incorrect)

16C

① (a) $v(t) = d'(t) = 4t$

(b) $v(2) = 4(2)$

$= 8\ m/s$

(c) $a = v'(t) = 4\ m/s^2$

② (a) $s(0) = (0)^3 - 5(0)^2 + 3(0) + 4$

$= 4m$

(b) $v(t) = s'(t)$

$= 3t^2 - 10t + 3$

$v(4) = 3(4)^2 - 10(4) + 3$

$= 11 m/s$

(c) $v(t) = 0$

$3t^2 - 10t + 3 = 0$

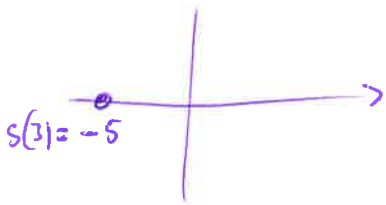
$(3t - 1)(t - 3) = 0$

$3t - 1 = 0 \quad t - 3 = 0$

$t = \frac{1}{3} s \quad t = 3s$

(d) $s(3) = (3)^3 - 5(3)^2 + 3(3) + 4$

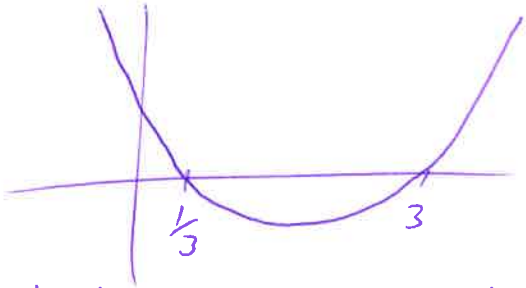
$= -5m$



the particle is 5m from the origin in the opposite direction to its initial position

16 C

2(e) $v(t) = 3t^2 - 10t + 3$



$-\frac{1}{3} < t < 3$

velocity is negative, moving back with respect to original direction

16C

3 (a) $v(t) = 8t^{\frac{1}{2}}$

$a(t) = v'(t) = 4t^{-\frac{1}{2}}$

$= \frac{4}{\sqrt{t}}$

$a(9) = \frac{4}{\sqrt{9}}$

$= \frac{4}{3} ms^{-1}$

(b) $a(t) = \frac{9}{16}$

$\frac{4}{\sqrt{t}} = \frac{9}{16}$

$4 \times 16 = 9\sqrt{t}$

$\frac{64}{9} = \sqrt{t}$

$t = \left(\frac{64}{9}\right)^2$

$t = 50.6$ seconds

16C

(4)

when $r=6$ $t=3s$ as moving at $2m/s$

$A(r) = \pi r^2$

$A'(r) = 2\pi r$

$A'(3) = 6\pi\ m/s$

(5) Extension question

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dV}{dr}(8) = 4\pi(8)^2$

$= 256\pi$

$\frac{dV}{dt} = 150$

$\frac{dV}{dr}\frac{dr}{dt} = 150$

$256\pi \frac{dr}{dt} = 150$

$\frac{dr}{dt} = \frac{150}{256\pi} = \frac{75}{128\pi}$